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On Lanczos' Algorithm for Tri-Diagonalization

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The Lanczos algorithm transforming a given matrix into a tri-diagonal form is well known in numerical analysis and is discussed in many literatures. The possibility of this algorithm is shown in Rutishauser's excellent paper [8]. However it seems to the author that no further theoretical consideration has been made since then.

The process starts from a pair of trial vectors x_1 and y_1. A pair of the i-th iterated vectors x_i and y_i can be constructed successively if $y_j{}^*x_j \neq 0$ $(1 \leqq j \leqq i-1)$. Hence, if $y_{p+1}{}^*x_{p+1}=0$ for some $p \leqq n-1$, we must modify the algorithm so as to continue. This is possible in case where $x_{p+1}=0$ or $y_{p+1}=0$, while any method of modification is not known in case where $x_{p+1} \neq 0$ and $y_{p+1} \neq 0$. We shall call the former case "lucky" and the latter "unlucky". The only thing for us to do in "unlucky" case is to choose new starting vectors x_1, y_1 and begin again in the hope that this case will not happen later. Rutishauser's result ([8] Satz 1) guarantees this possibility.

In practical computation, however, "unlucky" case may occur after repeated modifications in "lucky" cases. Once we encountered with "unlucky" case, we have to abandon all the efforts made before and start again with new trial vectors (if we stick to the old knowledge). Then a question arises naturally: Is it actually necessary to go back to the first step? In this paper we shall treat this problem. Roughly speaking, the answer is as follows: It is sufficient to go back to the latest modification. As a special case of this result, we can show that one of the initial vectors can be chosen arbitrarily to avoid "unlucky" case. Further it will be shown that there exists a vector x such that the algorithm starting from $x_1=y_1=x$ can be continued so that "unlucky" case may not occur. These results will be stated in Theorems 3-6 of §2 and a new procedure will be proposed at p. 279. Finally, in connection with the Lanczos algorithm, we shall give, in Appendix, some properties concerning the location of the eigenvalues of tri-diagonal matrices.

§1. Preliminaries

1.1. Let A be a given (complex or real) matrix of order n. Starting from a pair of initial vectors x_1 and y_1, construct a sequence of iterated vectors x_i, y_i as follows:

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