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Notes on Quasi-Valuation Rings

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Let R be a commutative ring with a unit. Then R is said to have a few zero-divisors if the total quotient ring of R has only a finite number of maximal ideals. Non zero-divisors in R are said to be regular elements in R, and an ideal A of R is called a regular ideal if A contains a regular element. Let R have a few zero-divisors and A a regular ideal of R. Then it is known that A is generated by regular elements in R(*). A ring R with a few zero-divisors is called a quasi-valuation ring if for any pair (a, b) of regular elements of R we have either $aR \subseteq bR$ or $bR \subseteq aR$. Let R be a quasi-valuation ring and M the ideal of R generated by all the non-unit regular elements in R. Then M is the unique regular maximal ideal of R unless every regular element is a unit in R.

In this paper we shall prove some properties of intersection of a finite number of quasi-valuation rings. Among others we have the following result:

Let $R_1, R_2, \cdots, R_n$ be quasi-valuation rings with the same total quotient ring K. Then $R = \bigcap_{i=1}^{n} R_i$ has K as the total quotient ring.

Let s be a non-unit regular element in R. If there exists a polynomial $f(X) = 1 + \sum_{i=1}^{m-1} h_i X^i + X^m$ with integer coefficients h_i such that $f(s)$ is a unit in R, we call $f(s)$ a unit associated to s.

LEMMA 1. *Let R be a semi-local ring with a unit (not necessary Noetherian) and let a be a non-unit regular element in R. Then there exists a unit $f(a)$ associated to a.*

PROOF. Let $M_1, M_2, \cdots, M_n$ be all the maximal ideals of R. Since a is not a unit and R has the identity, there is at least one maximal ideal containing a. We denote by A_1 the set of all the maximal ideals of R containing a. If $1+a$ is a unit, then $f(a) = 1+a$ is a unit associated to a. If $x_1 = 1+a$ is not a unit, then the set A_2 of all the maximal ideals of R which contain $1+a$ is not empty and $A_1 \cap A_2 = \phi$, and moreover, $0 \neq ax_1 \in M_i$ for all $M_i \in A_1 \cup A_2$. If $1+ax_1$ is a unit, then $f(a) = 1+ax_1$ is a unit associated to a.

If $x_2 = 1+ax_1$ is not a unit, then we proceed further as above. Since R is a semi-local ring, there exists a positive integer k such that $1+ax_1x_2\cdots x_k$ $(x_i = 1+ax_1x_2\cdots x_{i-1})$ is not contained in any one of maximal ideals M_i $(1 \leq i \leq n)$. Hence $f(a) = 1+ax_1x_2\cdots x_k$ is a unit associated to a.

(*) cf. E. D. DAVIS, *Overrings of commutative rings II*, Trans. Amer. Math. Soc. **110** (1964), 196-212.