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On the Fine Cauchy Problem for the System of Linear Partial Differential Equations

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In our previous paper [4, p. 406] we have introduced the notion of a canonical extension of a distribution in studying the distributional boundary values of holomorphic functions. The notion will be treated in this paper with considerable detail so as to be applied to the study of the fine Cauchy problem for the system of linear partial differential equations. Let Ω be a non-empty open subset $\subset R^n$ and T a positive number which may be $+\infty$. Let u be a distribution on $\Omega\times(0, T)$. We shall say that a distribution $\tilde{u}$ on $\Omega\times(-\infty, T)$ is a canonical extension of u if $\tilde{u}=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u$, where $\rho(t)$ is an arbitrary function with certain properties (cf. Definition 1 below) and $\rho_{(\varepsilon)}(t)=\rho\left(\frac{t}{\varepsilon}\right)$. If u happens to have the boundary value $\lim_{t\downarrow 0} u=\alpha$, then the identity:

$$\frac{\partial}{\partial t}(\rho_{(\varepsilon)}u)=\rho_{(\varepsilon)}\frac{\partial u}{\partial t}+\rho'_{(\varepsilon)}u$$

will imply that

$$\frac{\partial \tilde{u}}{\partial t}=\left(\widetilde{\frac{\partial u}{\partial t}}\right)+\alpha\otimes\delta_t.$$

The fact will be used to bring the initial conditions into the differential system as done in L. Schwartz [6, p. 133].

Section 1 is devoted to the discussions centering around the canonical extension $\tilde{u}$. In Section 2 we consider the canonical extension in a narrow sense and develop the same consideration as in Section 1. In Section 3 we deal with the fine Cauchy problem for the system of linear partial differential equations (cf. [6, p. 133]). For instance, consider the Cauchy problem for the system (we use the vector notation):

$$\frac{\partial u}{\partial t}=P(x, t, D_x)u+f$$

with the initial condition $\lim_{t\downarrow 0} u(x, t)=\alpha\in\mathcal{D}'(\Omega)$, where f is a given vector of distributions in $\mathcal{D}'(\Omega\times(0, T))$. Suppose f has the canonical extension $\tilde{f}$. We shall show in Theorem 1 below that to solve the Cauchy problem just considered is to find a vector v of distributions which satisfies the system: