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Modules which Have No Co-irreducible Submodules

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It is known that, for a ring R, every injective R-module has an indecomposable direct summand if and only if every ideal is an intersection of two ideals, at least one of which is irreducible ([4]). In [1], it is pointed out that the zero ideal of the ring of continuous functions defined on the interval [0, 1] does not satisfy the above condition and there are no other examples as far as the author knows.

The aim of this paper is to present such an example as a domain and investigate the characters of ideals which do not satisfy the condition. Also we shall settle several conjectures by making use of this example.

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Throughout this paper all rings will be commutative with unit and all modules will be unitary. For an ideal I and an element r of a ring R, $I: r$ means the ideal $\{s \in R; sr \in I\}$. For an element x of an R-module, $0(x)$ means the order ideal of x. We write $x \in S-T$ for $x \in S$ and $x \notin T$.

§1. Co-irreducible modules

Let R be a ring and M an R-module. We shall say that M is co-irreducible if $M \neq 0$ and for any non-zero submodules N_1 and N_2 of M, $N_1 \cap N_2 \neq 0$. If M is a co-irreducible R-module, then non-zero submodules and essential extensions of M are also co-irreducible. Let M be an R-module and N a submodule of M. We shall say that N is irreducible in M if M/N is co-irreducible. In other words, if $N = M_1 \cap M_2$ for submodules M_1 and M_2 of M, then $N = M_1$ or $N = M_2$. For an ideal I of R, we say that I is an irreducible ideal if I is irreducible in R as an R-module. Then prime ideals of R are irreducible.

Theorem 1.1. *The following conditions in a ring R are equivalent,*

1) *Any non-zero R-module contains a co-irreducible submodule.*

2) *If I is an ideal of R, different from R, then there exists an element r of R such that $I: r$ is an irreducible ideal.*

Proof. We assume the condition 1). Let $I (\subsetneqq R)$ be an ideal of R. Then the non-zero module $R/I = Rx$ contains a co-irreducible submodule Rrx for some $r \in R$. Since Rrx is isomorphic to $R/0(rx)$ and $0(rx) = I: r$, $I: r$ is irreducible. Conversely we assume the condition 2). It is sufficient to show the