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*A Note on the Space of Dirichlet-Finite Solutions of $\Delta u=Pu$ on a Riemann Surface**

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1. The space $PD(R)$ was initially investigated by Royden [10] and Nakai [7], with more recent contributions also by Nakai [8, 9] and Glasner-Nakai [2]. It was shown in [2] that the set Δ_P of P-energy nondensity points determines the space $PD(R)$ in some sense. In this note we give further evidence along these lines.

2. Let R be an open Riemann surface and $P\geq 0$, $P\not\equiv 0$ a density on R. Denote by $PD(R)$ the space of Dirichlet-finite C^2 solutions on R of the equation $\Delta u=Pu$. Let $\tilde{M}(R)$ be the class of all Dirichlet-finite Tonelli functions on R, and $\tilde{M}_\Delta(R)$ the set of functions $f\in\tilde{M}(R)$ such that $f=0$ on the Royden harmonic boundary Δ, of the Royden compactification R^*. Since $PD(R)\subset\tilde{M}(R)$, the orthogonal decomposition of $\tilde{M}(R)$ (cf. eg. Sario-Nakai [11]) yields a vector space isomorphism T: $PD(R)\rightarrow HD(R)$ which preserves the sup norm. The distribution of $PD(R)\mid\Delta$ in $HD(R)\mid\Delta$ is still an important subject for investigation (cf. Singer [12]).

We shall make essential use of the operator T_Ω given by

$$T_\Omega\phi=\frac{1}{2\pi}\int_\Omega G_\Omega(\cdot,\, z)\;\phi(z)P(z)\,dv(z),$$

where Ω is an open subset of R having a smooth relative boundary and $G_\Omega(\cdot,\, z)$ is the harmonic Green's function on Ω, $dv(z)=dx\,dy$. It is known that the Dirichlet integral of $T_\Omega u$ for $u\in PD(R)$ is given by

$$D_\Omega(T_\Omega u)=\frac{1}{2\pi}\int_{\Omega\times\Omega}G_\Omega(z,\, w)u(z)u(w)P(z)P(w)\,dv(z)\,dv(w).$$

For a comprehensive discussion of the operator T_Ω see Nakai [9]. A *P-energy nondensity point* z^* is a point of R^* with the property that there exists an open neighborhood U^* of z^* in R^* such that

$$\int_{U\times U}G_U(z,\, w)P(z)P(w)\,dv(z)\,dv(w)<\infty, \tag{1}$$

* Similar results have been obtained independently by Professor Wellington H. Ow, "*PD*-minimal solutions of $\Delta u=Pu$ on open Riemann surfaces", to appear in the Proc. Amer. Math. Soc.