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Characterizations of Radicals of Infinite Dimensional Lie Algebras

Dedicated to Professor Tôzirô Ogasawara
on the occasion of his retirement

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Introduction

Recently investigations have been made on the Lie algebras of infinite dimension. As the Lie analogues of the infinite group theory, B. Hartley [1] has considered the notions of subideals and ascendant subalgebras and studied the locally nilpotent radicals which reduce to the nilpotent radical in finite-dimensional case. In [4, 5] we have introduced and studied the locally solvable radicals which reduce to the solvable radical in finite-dimensional case. If $\mathfrak{X}$ is a coalescent (resp. an ascendantly coalescent) class of Lie algebras, for an arbitrary Lie algebra L we there defined the radical $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$) as the subalgebra generated by all the $\mathfrak{X}$ subideals (resp. all the ascendant $\mathfrak{X}$ subalgebras) of L. In particular, if the basic field is of characteristic 0, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-si}}(L)$ and $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-asc}}(L)$ are respectively the Baer radical $\beta(L)$ and the Gruenberg radical $\gamma(L)$ which are locally nilpotent [1], and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ are locally solvable radicals [4, 5], where $\mathfrak{N}$, $\mathfrak{S}$ and $\mathfrak{F}$ denote respectively the classes of nilpotent, solvable and finite-dimensional Lie algebras.

The purpose of this paper is to investigate the radicals of Lie algebras, especially to present certain characterizations of $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ and $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$ and to study two new radicals.

For a class $\mathfrak{X}$ of Lie algebras, we denoted by $\mathrm{L}\mathfrak{X}$ the collection of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subalgebra of L [4]. In Section 2, in connection with $\mathrm{L}\mathfrak{X}$ we define $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) as the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subideal (resp. an ascendant $\mathfrak{X}$ subalgebra) of L and study their properties. In Section 3 we show that if $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent), any Lie algebra L has a unique maximal $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideal (Theorem 3.2) and $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$) is the subalgebra generated by all the $\mathrm{M}\mathfrak{X}$ subideals (resp. all the ascendant $\acute{\mathrm{M}}\mathfrak{X}$ subalgebras) of L and belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) (Theorem 3.5). Hence if furthermore $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$) is an ideal of L then it is the unique