HIROSHIMA MATH. J.
3 (1973), 367–386

A Semigroup Treatment of the Hamilton-Jacobi Equation in One Space Variable

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(Received May 11, 1973)

1. Introduction

This paper has been motivated by a recent paper [2] by M. G. Crandall, in which the Cauchy problem for the first order quasilinear equation

$$(*) \qquad u_t+\sum_{i=1}^{n}(\phi_i(u))_{x_i}=0, \qquad x\in R^n,\ t>0,$$

is treated from the point of view of the theory of semigroups of nonlinear transformations. Crandall chose $L^1(R^n)$ as the Banach space associated with the Cauchy problem for $(*)$ and succeeded in constructing a semigroup of contractions in $L^1(R^n)$, which provides generalized solutions of the Cauchy problem in the sense of Kružkov [6] if the initial conditions lie in $L^1(R^n)\cap L^\infty(R^n)$.

In this paper we intend to treat the Cauchy problem (hereafter called (CP)) for the Hamilton-Jacobi equation

$$(\mathrm{DE}) \qquad u_t+f(u_x)=0, \qquad -\infty<x<\infty, \quad t>0,$$

from the same point of view. In our (CP), however, we shall, suggested by Kružkov [6], choose $L^\infty(R)$ as the Banach space which may be associated with it. As we shall see, the semigroup approach enables us to treat (CP) under the assumption that $f: R\to R$ is merely continuous. Moreover, as an intermediate step in the development, the existence and uniqueness of certain bounded (possibly generalized) solutions are established for the equation

$$(1) \qquad u+f(u_x)=h, \qquad -\infty<x<\infty,$$

for given h.

When $n=1$, there is clearly an intimate relationship between generalized solutions (cf. [6]) of the Cauchy problem for the quasilinear equation $(*)$ and the Hamilton-Jacobi equation (DE): If u is a generalized solution of the latter equation, then $v=u_x$ is a generalized solution of the former, and the converse is true. In this connection it is easy to see that when applied to (CP), Crandall's result can afford a semigroup of contractions on the subspace of $L^\infty(R)$ consisting of all continuous functions u such that both $\lim_{x\to\infty} u$ and $\lim_{x\to-\infty} u$ exist and are