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A Finite-Difference Method on a Riemann Surface

Hisao MIZUMOTO

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Introduction.

In the present paper we aim to discuss a method of finite-differences from the point of view of applications to the function theory. Since we speak of harmonic and analytic differentials and functions on a Riemann surface, we need to construct a theory of finite-differences on a *Polyhedron*.

Let u be a function defined at the points of the complex plane whose coordinates are integers. As a definition of a discrete harmonic function u on a plane, the so-called five-point formula

$$u(z+1)+u(z+i)+u(z-1)+u(z-i)-4u(z)=0$$

is generally used. How we define the conjugate discrete harmonic function and a discrete analytic function so that their definition match with the above definition of a discrete harmonic function, is an important problem. It is desirable, based on the definitions, to construct a theory of rich contents of discrete harmonic and analytic functions. As the works with this intention, we can mention Blanc [3], Lelong-Ferrand [10], [11], Isaacs [7], [8], Duffin [5], Hundhausen [6], etc.

Blanc [3] introduced the concepts of a *rèseau Riemannien* and a *rèseau conjuguè* on a plane, and introduced very general definitions of a discrete harmonic function and its conjugate function. He developed an interesting analogy with the type problem of a Riemann surface, and also he [2] developed an analogy with Nevanlinna's first and second fundamental theorems. However, it seems that he did not intend to make an effective use of a conjugate harmonic function on a rèseau conjuguè. Our definitions of a harmonic function and its conjugate function are similar to Blanc's.

Let f be a complex-valued function defined at the points of the complex plane whose coordinates are integers. Then Lelong-Ferrand [10], [11] introduced the following definition of a discrete analytic function f:

$$\frac{f(z+1+i)-f(z)}{1+i}=\frac{f(z+i)-f(z+1)}{i-1}. \tag{1}$$

If we set $f=u+iu^*$ where u and u^* are real, then it is seen that the discrete analyticity of f implies that u and u^* are discrete harmonic and satisfy a pair of difference equations which are analogous to the Cauchy-Riemann equations. She developed