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Character Groups of Toral Lie Algebras

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Introduction

It is well known that the duality and categorical equivalence hold between algebraic tori and character groups (e.g., [1], ch. III). In this paper we develop an analogy for Lie algebras. General properties of toral Lie algebras are stated in [3] and [5]. Their characters are introduced by Seligman in [3] and applied to algebraic Lie algebras in [3] and [4].

Let T be a toral Lie algebra and let $X(T)$ be the character group of T. Then it is proved that the (contravariant) functor $X: T \mapsto X(T)$ is actually an equivalence of categories (Theorem 1) and in this relation every subalgebra (resp. quotient algebra) of T corresponds to a quotient group (resp. subgroup) of $X(T)$ (Proposition 3).

As an application we generalize some of the results in [3]. Namely, if T satisfies a certain condition which generalizes that the base field is finite then the properties (a) and (b) of Theorem 7 in [3] are equivalent (Theorem 2) and the direct sum decomposition of T as in Theorem 8 in [3] holds (Theorem 3).

The main tools of the paper are the rationality property for vector spaces in terms of Galois groups which is described in [1] and the direct sum decomposition, stated in [2], of a vector space on which a nilpotent Lie algebra acts.

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1. Preliminaries and notations

Let k be any field of characteristic $p>0$. Let L be a Lie p-algebra over k of finite dimension with a p-map $x \mapsto x^p$. An element $x \in L$ is said to be separable if x is represented as a linear combination of x^p, $x^{p^2}, \dots$. If T is an abelian Lie p-algebra over k and every element of T is separable then T is called a torus or a toral Lie algebra over k. Some criteria for tori are found in [5]. Cleary every (p-)subalgebra of a torus is itself a torus. In this paper homomorphisms of Lie p-algebras always mean Lie algebra homomorphisms which are compatible with p-maps.

Let $\bar{k}$ be the algebraic closure of k and k_s be the separable closure of k in $\bar{k}$. Then $\bar{k}$ and k_s are regarded as Lie p-algebras over k with natural p-th power.