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An Inequality for Certain Functional of Multidimensional Probability Distributions

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§ 1. Introduction and the results

Denote by $\mathscr{P}$ the class of all probability distributions f in R^d such that $\int |x|^2 f(dx) < \infty$ and $\int (x_i - \mu_i)^2 f(dx) > 0\ (1 \leqq i \leqq d)$, where $\mu = (\mu_1, \dots, \mu_d)$ is the mean vector of f. For each $f \in \mathscr{P}$, denote by g_f the Gaussian distribution with the same mean vector and variance matrix as those of f. We introduce a functional $\mathfrak{e}$ on $\mathscr{P}$ by

$$\mathfrak{e}[f] = \inf E\{|X - Y|^2\}, \qquad f \in \mathscr{P},$$

where the infimum is taken over all pairs of R^d-valued random variables X and Y defined on a probability space $(\Omega, \mathscr{F}, P)$ and distributed according to f and g_f respectively. We also write $\mathfrak{e}[X]$ for $\mathfrak{e}[f_X]$, where f_X is the probability distribution of a random variable X.

In the one dimensional case, the functional $\mathfrak{e}$ was introduced and its basic properties were studied in [4] with an application to Kac's one-dimensional model of a Maxwellian gas. The purpose of this paper is to extend some results in [4] to the multi-dimensional case, that is, we will prove the following theorems.

THEOREM 1. *Let X and Y be random variables with probability distributions $f \in \mathscr{P}$ and g_f respectively, and assume that $\mathfrak{e}[f] = E\{|X - Y|^2\}$. Then, X is equal to some Borel function of Y almost surely.*

THEOREM 2. *Let X_1 and X_2 be independent random variables with probability distributions belonging to $\mathscr{P}$. Then,*

$$\mathfrak{e}[X_1 + X_2] < \mathfrak{e}[X_1] + \mathfrak{e}[X_2]$$

unless both X_1 and X_2 are Gaussian. In other words, the functional equation

$$\mathfrak{e}[f_1 * f_2] = \mathfrak{e}[f_1] + \mathfrak{e}[f_2], \qquad f_1, f_2 \in \mathscr{P}$$

gives a characterization of Gaussian distributions.