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On Characterizations of Dedekind Domains

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Introduction

Let R be a commutative ring with unit element and M be an R-module. We consider the following two properties on M.

$(P)_R$: If $M=N_1+N_2$ where N_1 and N_2 are R-submodules of M, then $N_1=M$ or $N_2=M$.

$(Q)_R$: if N_1 and N_2 are R-submodules of M, then $N_1 \supset N_2$ or $N_2 \supset N_1$.

Clearly, the property $(Q)_R$ implies the property $(P)_R$ and the property $(P)_R$ implies that M is indecomposable.

In the case that R is a Dedekind domain, we shall exhibit all R-modules which satisfy $(P)_R$ and at once see that they satisfy also $(Q)_R$.

If we restrict ourselves to abelian groups, those groups which satisfy $(P)_Z$ are subgroups of $Z(p^\infty)$ for some prime p. In fact, this result came first and then it has been generalized to modules over a Dedekind domain R.

Next, suppose R to be a noetherian integral domain such that if $(P)_R$ is satisfied by an R-module M then so is $(Q)_R$. We shall show R must be a Dedekind domain if $R_\mathfrak{p}$ is analytically irreducible for any maximal ideal $\mathfrak{p}$. This gives us a new characterization of Dedekind domains.

Finally, in §2, we shall discuss a relation between the notion of purity and that of essentiality and get another characterization of Dedekind domains.

§1. In the following we denote by $E_R(M)$ the injective envelope of an R-module M. First we determine all R-modules which satisfy the property $(P)_R$, or equivalently the property $(Q)_R$, when R is a Dedekind domain.

THEOREM 1. *Let R be a Dedekind domain, K be its quotient field and M be an R-module. Then the following statements are equivalent:*

(1) *M has $(P)_R$.*

(2) *M has $(Q)_R$.*

(3) *If R is not a discrete valuation ring, then M is isomorphic to a submodule of $E_R(R/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$ in R. If R is a discrete valuation ring, M is isomorphic to R, K or a submodule of $E_R(R/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$.*