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Eigenfunctions of the Laplacian on a Hermitian Hyperbolic Space

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Let G be a connected real semisimple Lie group of real rank one with finite center, K a maximal compact subgroup, $G=KAN$ an Iwasawa decomposition and M the centralizer of A in K. We put $X=G/K$ and $B=K/M$. Let Δ denote the laplacian on X corresponding to the G-invariant riemannian metric on X induced by the Killing form of the Lie algebra of G. In [2, Chap. IV, Th. 1.8], S. Helgason proved that when $G=SU(1, 1)$, any eigenfunction of Δ can be given as the Poisson transform of a (Sato's) hyperfunction on B, and suggested the possibility of generalizing the theorem to the case of a (non-compact) symmetric space of rank one, which we shall call Helgason's conjecture.

The purpose of this paper is to prove that when X is a hermitian hyperbolic space $SU(n, 1)/S(U_n\times U_1)$, Helgason's conjecture is valid in a weak sense. That is, any eigenfunction of Δ with real eigenvalue $\mu\geqq-<\rho, \rho>$ can be given as the Poisson transform of a hyperfunction on B (Corollary 4.5). For a real hyperbolic space $SO_0(n, 1)/SO(n)$, the author proved in [7] that Helgason's conjecture is valid for any complex eigenvalue.

The construction of this paper is as follows. In §1, we define the Poisson transform of a continuous function and state some results on this transform. In §2, we review the structure of the Lie algebra $\mathfrak{su}(n, 1)$ and investigate the eigenvalues of some differential operators. In §3, the Poisson transform of a K-finite function on B are determined explicitly. In the final section, by using the results in §3 we prove that for $s\geqq 0$, Poisson transform $\mathscr{P}_s$ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{H}_s(X)$ (Theorem 4.4), where $\mathscr{B}(B)$ is the space of hyperfunctions on B and $\mathscr{H}_s(X)$ is the space of eigenfunctions of Δ with eigenvalue $(s^2-1)<\rho, \rho>$. From this theorem Corollary 4.5 follows immediately.

We shall use the standard notation $\boldsymbol{N}$, $\boldsymbol{R}$, $\boldsymbol{C}$ for the set of natural numbers, the field of real numbers and the field of complex numbers respectively; $\boldsymbol{N}^0$ is the set of non-negative integers. If E is a differentiable manifold, $C(E)$ (resp. $C^\infty(E)$) denotes the space of all continuous (resp. smooth) functions on E.

§1. Poisson transform and its fundamental properties

In this section, we define the Posison transform and gather some results on