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A Note on Graded Gorenstein Modules

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Recently the following conjecture was proposed by M. Nagata in [6]. Let $A=\sum_{n\geqq 0} A_n$ be a commutative Noetherian graded ring. If $A_{\mathfrak{m}}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$ with $\mathfrak{m}\supset \sum_{n\geqq 1} A_n$, then A is Cohen-Macaulay. This conjecture was solved affirmatively by J. Matijevic and P. Roberts in [5]. The aim of this paper is to prove the following theorem which generalizes the assertion in [5].

THEOREM. *Let $A=\sum_{n\in Z} A_n$ be a commutative Noetherian graded ring and $M=\sum_{n\in Z} M_n$ be a non-zero, finite graded A-module. If $M_{\mathfrak{p}}$ is a Gorenstein $A_{\mathfrak{p}}$-module (resp, a Cohen-Macaulay $A_{\mathfrak{p}}$-module) for every homogeneous prime ideal $\mathfrak{p}\in \mathrm{Supp}(M)$, then M is Gorenstein (resp. Cohen-Macaulay).*

1. We denote by $\mu^i(\mathfrak{p}, M)$ the dimension of the $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$-vector space $\mathrm{Ext}^i_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}})$ (cf. [1]) and by $ht_M\mathfrak{p}$ the Krull dimension of the local ring $A_{\mathfrak{p}}/Ann(M)A_{\mathfrak{p}}$ (cf. [7]), where $Ann(M)$ is the annihilator of M and $\mathfrak{p}\in \mathrm{Supp}(M)$. The following lemma, due to Bass and Sharp, plays an important role in our discussion.

LEMMA 1 (Bass [1, (3.7)] and Sharp [7, (3.11)]). *Let M be a finite A-module.*
(i) *M is a Cohen-Macaulay module if and only if, for each $\mathfrak{p}\in \mathrm{Supp}(M)$, $\mu^i(\mathfrak{p}, M)=0$ whenever $i<ht_M\mathfrak{p}$.*
(ii) *The following conditions are equivalent.*
  (1) *M is a Gorenstein module.*
  (2) *For each $\mathfrak{p}\in \mathrm{Supp}(M)$, $\mu^i(\mathfrak{p}, M)=0$ if and only if $i\neq ht_M\mathfrak{p}$.*

For an ideal $\mathfrak{a}$ of the graded ring A we let $\mathfrak{a}^*$ denote the homogeneous ideal generated by homogeneous elements of $\mathfrak{a}$.

LEMMA 2. *Let M be a graded A-module and $\mathfrak{p}$ a prime ideal of A. Then $\mathfrak{p}\in \mathrm{Supp}(M)$ if and only if $\mathfrak{p}^*\in \mathrm{Supp}(M)$.*

PROOF. Suppose that $M_{\mathfrak{p}}=0$; then, for each homogeneous element m in M, there is a homogeneous component of s with $sm=0$, say s_u, which is not contained in $\mathfrak{p}$. Clearly $s_um=0$ and this implies $M_{\mathfrak{p}^*}=0$. The converse is obvious. q.e.d.