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Extremum Problems on an Infinite Network

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Introduction

Network problems are discussed usually on a finite graph. Duffin [5] investigated the extremal length of a network on a finite graph and suggested a relation between potential theory and network theory. Derrick [4] and Ohtsuka [6] generalized Duffin's results to the continuous case without using network theory.

We shall study in this paper the extremal length of a network on an infinite graph which has a countably infinite number of nodes and arcs. We use some techniques which are standard in potential theory (for instance [1], [2] and [3]) and go along Duffin's arguments.

Some definitions and notations related to network theory are given in § 1. The extremal length of a network is studied in § 4 with the aid of the functional spaces defined in § 2 and the fact in § 3 that max-potential equals min-work. The duality relation between the max-flow problem and the min-cut problem, which is investigated in § 6, does not hold in general for infinite linear programming problems. We shall treat three kinds of the extremal widths of a network in § 7 by using some results in § 5 and § 6. The reciprocal relation between the extremal length and one of the extremal widths is also studied in § 7. We shall be concerned with Duffin's path-cut inequality in § 8.

§ 1. Notations and network definitions

A graph is intuitively a geometric figure consisting of points (which we shall call nodes) and line segments (which we shall call arcs) connecting a node to another. To each arc we assign a direction. Denote by X the set of nodes and by Y the set of arcs. Since we always consider the case where X and Y consist of a countably infinite number of elements, we put for simplicity

$$X = \{0, 1, 2, \ldots, n, \ldots\},$$

$$Y = \{1, 2, \ldots, n, \ldots\}.$$

Define the node-arc incidence matrix $K = (K_{vj})$ by $K_{vj} = 1$ if arc j is directed toward node v, $K_{vj} = -1$ if arc j is directed away from node v and $K_{vj} = 0$ if arc j and node