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Parallelizability of Grassmann Manifolds

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§1. Introduction

Let $G_{n,m}$ be the Grassmann manifold of all m-planes through the origin of the Euclidean n-space R^n. A. Neifahs [3] proved that n is a power of 2 if $G_{n,m}$ is parallelizable.

In this note, we prove the following

THEOREM 1.1. *$G_{n,m}$ is parallelizable, i.e., the tangent bundle of $G_{n,m}$ is trivial, if and only if*

$$n = 2, 4 \text{ or } 8; \qquad m = 1 \text{ or } n-1.$$

To prove this theorem, we use the following theorem.

For a real vector bundle ξ, we denote by $Span\,\xi$ the maximum number of linearly independent cross-sections of ξ. Especially, we denote $Span\,M = Span\,\tau M$, where τM is the tangent bundle of a C^∞-manifold M.

THEOREM 1.2. *Let ξ_k be the canonical line bundle over the real projective k-space RP^k, and $n\xi_k$ the Whitney sum of n-copies of it.*

Then, $Span\,G_{n,m} \geqq k$ implies $Span\; nm\xi_{n-m} \geqq m^2+k$.

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§2. Proof of Theorem 1.2

Let $\gamma_{n,m}$ be the canonical m-plane bundle over $G_{n,m}$, i.e., the total space of $\gamma_{n,m}$ be the subspace of $G_{n,m} \times R^n$ consisting of all pairs (x, v) where $x \in G_{n,m}$ and v is a vector in x. Then, by [2, Problem 5–B],

$$\tau G_{n,m} \cong \mathrm{Hom}(\gamma_{n,m}, \gamma_{n,m}^{\perp}), \tag{2.1}$$

where $\gamma_{n,m}^{\perp}$ denotes the orthogonal complement of $\gamma_{n,m}$ in the trivial bundle $G_{n,m} \times R^n \to G_{n,m}$.

Consider the Stiefel manifold $V_{n,m}$ of all orthonormal m-frames in R^n, which has the involution by sending each $(v_1,\ldots, v_m)$ to $(-v_1,\ldots, -v_m)$. By [5, Prop. 1], we see the following fact.