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Geometry of Homogeneous Lie Loops

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Introduction

Let M be a differentiable manifold with a given linear connection ∇ and $e \in M$ be a fixed point. Then we have considered in [5] the local multiplication μ at e compatible with ∇, which is given by

$$\mu(x, y) = \mathrm{Exp}_x \circ \tau_{e,x} \circ \mathrm{Exp}_e^{-1}(y),$$

where Exp_x denotes the exponential mapping at x and $\tau_{e,x}$ denotes the parallel displacement of tangent vectors along the geodesic joining e to x in a normal neighborhood of e.

If M is a reductive homogeneous space A/K with the canonical connection, due to K. Nomizu, then the local multiplication μ given above satisfies

$$\mu(x, y) = (\exp X)\cdot y;\ X = \mathrm{Exp}_e^{-1}(x) \in \mathfrak{M} \subset \mathfrak{A},$$

where $\mathfrak{A} = \mathfrak{M} + \mathfrak{K}$ is the decomposition of the Lie algebra of A such that $\mathrm{ad}\,(K)\mathfrak{M} \subset \mathfrak{M}$. (Cf. [15, Theorem 10.2].) Therefore, if M is reduced to a Lie group A itself, then the canonical connection is reduced to the $(-)$-connection of [3] and the local multiplication μ coincides with the multiplication of A in local.

These facts suggest us a problem of the existence of a global differentiable binary system on a reductive homogeneous space A/K, which coincides locally with the above geodesic local multiplication μ. We have been interested in this problem and in the question how such a multiplication relates to the canonical connection and to the general Lie triple system defined on the tangent space $\mathfrak{M}$, which will be called the Lie triple algebra in this paper (cf. [5–8]).

The main purpose of the present paper is to investigate the above problem and to provide the basic concepts to construct the global theory of differentiable binary systems, as an analogy and also as a generalization of the theory of Lie groups and Lie algebras.

Our considerations are based on the purely algebraic concept of a *homogeneous loop* (Definition 1.4) and the concept of a *homogeneous Lie loop* (Definition 3.1), a homogeneous loop admitting a natural differentiable structure. We shall prove that any homogeneous Lie loop has the canonical connection and is a reductive homogeneous space. We shall investigate the condition for the multiplication of such a loop G to coincide in local with the geodesic local multi-