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On Extremal Sets of Parallel Slits

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Introduction

Let W be a region in the extended z-plane containing the point ∞ and let $\{W_n\}_{n=1}^{\infty}$ be a regular exhaustion of W containing the point ∞, i.e., let W_n be regions such that $\infty \in W_n$, $\overline{W_n} \subset W_{n+1}$, $\cup W_n = W$ and the boundary of each W_n consists of a finite number of disjoint analytic Jordan curves. Let P_n be the unique vertical slit mapping of W_n with the following expansion about ∞:

$$P_n(z) = z + \frac{a_{1,n}}{z} + \cdots .$$

D. Hilbert, P. Koebe and R. Courant showed that P_n converges uniformly on compact subsets of W to a vertical slit mapping P_W, i.e., every component of the boundary of $P_W(W)$ is either a point or a line segment parallel to the imaginary axis. Let $\mathfrak{F}$ be the family of univalent meromorphic functions F on W with the expansion

$$(*) \qquad F(z) = z + \frac{a_1(F)}{z} + \cdots \qquad \text{about } \infty .$$

Then P_W is the unique function minimizing Re $a_1(F)$ in $\mathfrak{F}$.

P. Koebe [4] showed that the complement $(P_W(W))^c$ of $P_W(W)$ has vanishing area. Therefore, for a region of infinite connectivity, the uniqueness of vertical slit mapping with the expansion $(*)$ does not always hold. In 1918, P. Koebe [5] called $P_W(W)$ the minimal vertical slits region. For an arbitrary plane region W containing ∞, the univalent meromorphic mapping of W with the expansion $(*)$ onto a minimal vertical slits region is uniquely determined. In the present paper we shall study the complements of minimal vertical slits regions. We call them extremal sets of vertical slits and denote their class by $\mathscr{E}$. P. Koebe [5] obtained the following results:

(I) E is a set of class $\mathscr{E}$ if and only if E is a bounded closed set such that $\int_{E^c} \partial f/\partial y \, dxdy = 0$ for every $f \in \mathbf{M}(E^c)$ which vanishes identically on a neighborhood of ∞, where $\mathbf{M}(E^c)$ denotes the class of Royden functions on E^c (see § 2).

(i) Every set of class $\mathscr{E}$ has vanishing area.