HIROSHIMA MATH. J.
5 (1975), 473–485

Some Counterexamples Related to Prime Chains in Integral Domains

Kazunori FUJITA

(Received May 19, 1975)

In this paper all rings are assumed to be commutative with identity. If A is a noetherian Hilbert ring which satisfies the second chain condition for prime ideals, then the polynomial ring $A[X]$ in an indeterminate X over A has the second chain condition for prime ideals ([11], Theorem 1.14). However, in Section 1, we show that $A[X]$ does not necessarily satisfy the first chain condition for prime ideals, even though A is a noetherian Hilbert ring which satisfies the first chain condition for prime ideals. If a ring A satisfies the first chain condition for prime ideals, then as we know, for each prime ideal $\mathfrak{p}$ in A, $ht(\mathfrak{p})+\dim(A/\mathfrak{p})=\dim(A)$. However, it is unknown whether the converse of this statement is true or not ([7], Remark 2.25). Moreover, in Section 1, we give a noetherian integral domain such that the converse is false. Let A be a noetherian semi local ring such that $ht(\mathfrak{p})+\dim(A/\mathfrak{p})=\dim(A)$ for any non maximal prime ideal $\mathfrak{p}$ in A. Then it is known that $ht(\mathfrak{m})=\dim(A)$ or $ht(\mathfrak{m})=1$ for any maximal ideal $\mathfrak{m}$ in A. But it is unknown whether this assertion is true or not for a general noetherian ring ([7], Remark 2.6). In Section 2, we give a noetherian integral domain such that the above assertion is false. This example shows besides that the statement b) and the statement c) of Remark 2.25 of Ratliff's paper [7] are not equivalent: Even if $\dim(A/\mathfrak{p})=\dim(A)-1$ for each height one prime ideal $\mathfrak{p}$ in a noetherian integral domain A, the equality $ht(\mathfrak{P})+\dim(A/\mathfrak{P})=\dim(A)$ does not necessarily hold for any prime ideal $\mathfrak{P}$ in A. In Section 3, making use of the example given in Section 2, we construct a non-catenarian local integral domain D such that for each height one prime ideal $\mathfrak{p}$ in D, $ht(\mathfrak{p})+\dim(D/\mathfrak{p})=\dim(D)$ (cf. [9], p. 232).

Throughout this paper the notation $M\subset N$ (or $N\supset M$) means that M is a proper subset of N.

The author wishes to express his gratitude to Professor H. Yanagihara for his valuable advice and his comments in writing this paper.

1. It is known that if a ring A satisfies the first chain condition for prime ideals, then for each prime ideal $\mathfrak{p}$ in A, $ht(\mathfrak{p})+\dim(A/\mathfrak{p})=\dim(A)$ ([7], p. 1083). Moreover, in [8], Ratliff proved that if A is a noetherian local domain, then the converse of this assertion holds. However it is an open problem whether or not the converse holds in general case ([7], p. 1085). The purpose of this section