HIROSHIMA MATH. J.
6 (1976), 171–181

Classification of Free Involutions on Surfaces

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(Received September 8, 1975)

§1. Introduction

A (continuous) map $\alpha: M \to M$ of a space M into itself is called an involution if $\alpha^2 = id$. We say that an involution α on M is equivalent to an involution α' on M' if there exists a homeomorphism $h: M \approx M'$ such that $\alpha' h = h\alpha$. The purpose of this note is to classify (fixed point) free invoultions on compact connected surfaces by this equivalence relation.

For an involution α on M, we obtain its orbit space M/α from M by identifying x with $\alpha(x)$ for $x \in M$. Then, we have the following

THEOREM 1.1. *Assume that X is a compact connected surface of genus g and the boundary ∂X consists of l components. Then the number n of equivalence classes of free involutions on connected surfaces, whose orbit spaces are homeomorphic to X, is given by*

$$n = \begin{cases} [l/2] + \min\{g, 1\} & \text{if } X \text{ is orientable,} \\ [l/2] + \min\{g, 3\} & \text{if } X \text{ is non-orientable.} \end{cases}$$

Now, we use the following notation:

(1.2) Let $\alpha: M \to M$ be an involution on a surface M of genus g such that the boundary ∂M has l components and the number of α invariant components is l_0 ($\leqq l$). Then, *the type* of such α is $(g, l, l_0, 1)$ if M is orientable and α preserves the orientation, $(g, l, l_0, -1)$ if M is orientable and α reverses the orientation, and $(g, l, l_0, 0)$ if M is non-orientable.

Then we have the following classification theorem of free involutions on compact connected surfaces.

THEOREM 1.3. (i) *There exists a free involution of type (g, l, l_0, ε) if and only if we have the following* (I), (II) *or* (III):

(I) *$\varepsilon = 1$, $l_0 \geqq 0$ is even, $l \geqq l_0$ is even and $g + 2\min\{l_0, 1\} - l_0/2 \geqq 1$ is odd;*

(II) *$\varepsilon = -1$, $l_0 = 0$, $l \geqq 0$ is even and $g \geqq 0$;*

(III) *$\varepsilon = 0$, $l_0 \geqq 0$ is even, $l \geqq l_0$ is even and $g + 2\min\{l_0, 1\} - l_0 \geqq 2$ is even.*

(ii) *There exist two free involutions of type (g, l, l_0, ε) up to equivalence if $\varepsilon = l_0 = 0$, $l \geqq 0$ is even and $g \geqq 4$ is even, and otherwise a free involution of type (g, l, l_0, ε) is unique up to equivalence.*