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On the Fourier Transform of Rapidly Decreasing Functions of L^p Type on a Symmetric Space

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1. Introduction

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let $G=KAN$ be a fixed Iwasawa decomposition and M the centralizer of A in K. In a series of his papers Harish-Chandra introduced the Schwartz space $\mathscr{C}(G)$, in analogy to the space $\mathscr{S}(\boldsymbol{R}^n)$, of rapidly decreasing functions on the real euclidean space $\boldsymbol{R}^n$([10]), and also as one of the family of the whole spaces $\mathscr{C}^p(G)$. It is a problem to know whether one can carry out a Fourier analysis of the member of $\mathscr{C}^p(G)$ and know the image of $\mathscr{C}^p(G)$ by the Fourier transform, when possible.

After Harish-Chandra, Eguchi-Okamoto [3] introduced the Schwartz space $\mathscr{C}(G/K)$ on the symmetric space G/K, which is a subspace of the space $\mathscr{C}(G)$, and characterized the image of it by the Fourier transform. In this paper we consider the Fourier transform of the subspaces $\mathscr{C}^p(G/K)$ $(0<p<2;\ \mathscr{C}^2(G/K)=\mathscr{C}(G/K))$ consisting of functions in $\mathscr{C}^p(G)$ which are invariant under right K action.

Let $0<p<2$. Then the space $\mathscr{C}^p(G/K)$ is contained in $\mathscr{C}(G/K)$ and so, for any $f\in\mathscr{C}^p(G/K)$ its Fourier transform $\tilde{f}$ is defined. For a general element $f\in\mathscr{C}(G/K)$, $\tilde{f}$ is a C^∞ function on $\mathfrak{a}^*\times K/M$ with a growth condition and a property of symmetry; but if f is an element of $\mathscr{C}^p(G/K)$, $\tilde{f}$ extends analytically to the interior of a tubular domain with respect to the first component. We denote the tubular domain by F^p. The main theorem of this paper is that the space $\mathscr{Z}(F^p\times K/M)$ consisting of these functions which have holomorphic extension to $\operatorname{Int}F^p$ and such symmetry and growth, is the just image of the Fourier transform of $\mathscr{C}^p(G/K)$ in real rank one case.

A brief sketch of the proof of surjectivity is as follows: Let $\hat{K}^0$ be the set of the equivalence classes of unitary representations of K which are class 1 with respect to M. Let φ be a function in $\mathscr{Z}(F^p\times K/M)$ and f be the Fourier inverse image of φ. Applying the theorem for the Fourier transform of smooth functions on K/M (Sugiura [11]), we obtain a family of functions $\varphi^\delta(\delta\in\hat{K}^0)$ with values in endomorphisms of the representation space of δ. Then φ^δ has a growth with respect to δ. From this and the fact that f is the sum of trace of inverse image f^δ of φ^δ, it follows that $f\in\mathscr{C}^p(G/K)$. In order to show that f^δ satisfies the