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A Note on Coalgebras and Rational Modules

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1. Introduction

Let C be a coalgebra and C* its dual algebra. Every right C-comodule is equipped functorially with a left C*-module structure. A left C*-module thus obtained is called rational. On the other hand every left C*-module M has a unique maximal rational submodule M^{rat} and the correspondence $M \mapsto M^{rat}$ is a functor from the category of left C*-modules to that of rational ones which form a full subcategory of the former. This functor is left exact.

In this note we study the relation between the structure of a coalgebra C and the functor $M \mapsto M^{rat}$.

In Section 3 we consider the exactness of the functor and show the following: When C is irreducible, the functor is exact if and only if C is of finite dimension. When C is cosemisimple, the functor is exact. When C is cocommutative, the functor is exact if and only if C is a direct sum of finite-dimensional subcoalgebras.

In [3] Radford has proved that if every open left ideal in C* is finitely generated, then the class of rational modules is closed under group extensions. And recently Lin [2] investigated as an application of the torsion theories the structure of a coalgebra for which the functor is a left exact radical, i.e., the class of rational modules is closed under group extensions. In Section 4 we study the extension problem and prove the converse of the Radford's result above when the coalgebra has a finite-dimensional coradical or when the coalgebra is cocommutative (Theorem 4.6 and Corollary 4.9). We don't use the torsion theories but some topological concepts in [3].

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2. Preliminaries.

Throughout this note, vector spaces, coalgebras and algebras we consider are all over a fixed commutative field k and all linear mappings are k-linear. We follow the terminology in [4] with a few exceptions.

(2.1) Let £ be a vector space and E^* its dual space. E^* has the weak-*topology. An open subspace of E^* with this topology is just a closed and