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Integrability of Nonoscillatory Solutions of a Delay Differential Equation

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The purpose of this note is to extend Dahiya and Singh's result [1, Theorem 4] to n-th order equations and give a straightforward proof of the theorem. Examples for $n=2$ which are not covered by Theorem 4 [1] will also be given.

We are concerned with the n-th order delay equation

$$(r(t)y^{(n-1)}(t))'+(b(t)y(t))'+a(t)y(g(t))=f(t), \qquad n>2, \tag{1}$$

where the functions $r(t)$, fo(0>"(0> $g(t)$ and/(O arecontinuous on the whole real line.

A nontrivial solution of (1) which exists on $[t_0, \infty)$ is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is said to be *nonoscillatory.*

THEOREM. *Assume that*

(i) $a(t)\geq M$ *on* $[t_0, \infty)$ *for some constant* $M>0$,

(ii) fe(0>0<m $[t_0, \infty)$,

(iii) $g(t)\to\infty$ *as* $t\to\infty$ *and* $0\leq g'(t)\leq 1$ *on* $[t_0, \infty)$,

(iv) $r(t)>0$ *on* $[t_0, \infty)$ *and* $\int_{t_0}^{\infty}\frac{1}{r(t)}dt=\infty$,

and

(v) $\int_{t_0}^{\infty}|f(t)|dt<\infty$.

Then every nonoscillatory solution of (1) *on* $[t_0, \infty)$ *is integrable.*

PROOF. Let $y(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that there is a $T_0>t_0$ such that $y(t)>0$ for $t>T_0$. Since $g(t)\to\infty$ as $t\to\infty$, there is a $T_1>T_0$ such that $y(g(t))>0$ for $t>T_1$.

Integrating (1) from T_1 to $t>T_1$, we obtain

$$r(t)y^{(n-1)}(t)-r(T_1)y^{(n-1)}(T_1)+b(t)y(t)-b(T_1)y(T_1)+\int_{T_1}^{t}a(s)y(g(s))ds \tag{2}$$
$$\leq\int_{T_1}^{t}|f(s)|ds.$$

The lefthand side of (2) remains bounded as $t\to\infty$.

Suppose $\int_{T_1}^{\infty}y(g(t))dt=\infty$. Then $r(t)y^{(n-1)}(t)\to-\infty$ as $t\to\infty$. Condition