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A Semigroup Treatment of the Mixed Problem for the Hamilton-Jacobi Equation in One Space Variable

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1. Introduction

This paper deals with the mixed problem in the domain $D=\{(t, x);\ 0\leq t<+\infty,\ 0\leq x\leq 1\}$ for the Hamilton-Jacobi equation (hereafter called (MP))

$$u_t+f(u_x)=0, \qquad t>0, \quad 0<x<1, \tag{1.1}$$

$$u(0, x)=u_0(x), \qquad 0\leq x\leq 1, \tag{1.2}$$

$$u(t, 0)=u(t, 1)=0, \qquad t\geq 0. \tag{1.3}$$

This investigation was strongly motivated by a recent paper [1] by S. Aizawa, in which the Cauchy problem for the Hamilton-Jacobi equation

$$u_t+f(u_x)=0, \qquad t>0, \quad -\infty<x<\infty, \tag{$*$}$$

is treated from the viewpoint of the nonlinear semigroup theory. Aizawa succeeded in constructing a global solution of the Cauchy problem for ($*$) in the sense of Kružkov when the initial value lies in $W_1^\infty(R)$ and its derivative is continuous in $L^1(R)$ under the assumption that f is merely continuous.

Mixed problems for the Hamilton-Jacobi equation have been treated by several authors. See, E. D. Conway and E. Hopf [7], S. Aizawa and N. Kikuchi [3] and S. H. Benton [5, 6]. These authors proved the existence of generalized solutions of mixed problems by using the variational method under the assumption that f is convex.

The purpose of this paper is to prove the existence of a generalized solution of (MP) by using the nonlinear semigroup theory. Our treatment needs no convexity condition on f. Here we define a generalized solution of (MP) in the sense of Kružkov [13].

DEFINITION 1.1. *A Lipschitz continuous function* $u(t, x)$ *on* $D=[0, \infty)\times[0, 1]$ *is called a generalized solution of* (MP) *if*

(i) *u satisfies* (1.1) *a.e. on D as well as* (1.2) *and* (1.3),

(ii) *u satisfies the integral inequality*