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Balayage, Capacity and a Duality Theorem in Dirichlet Spaces

Takahide KUROKAWA

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Introduction

H. Cartan [2] systematically applied the method of Hilbert space to the study of capacity and balayage in the classical potential theory. His idea was generalized to the axiomatic theory of Dirichlet spaces by A. Beurling and J. Deny [1]. Balayages and capacities in Dirichlet spaces are studied in [1], [4], [5], [6] and [7] to some extent. In the present paper, we proceed to study inner and outer balayages and capacities in Dirichlet spaces. We shall show that characterizations of these notions are obtained as consequences of a certain duality theorem (Theorem 3.1). As an application, we shall show that the inner balayage and the outer balayage coincide for K-analytic sets.

§1. Cones and T-cones in a Hilbert space

In this section, let H be a real Hilbert space with norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)$. A cone in H is a set S in H such that $\lambda \geqq 0$ and $x \in S$ imply $\lambda x \in S$. A set S in H will be called a T-cone (T stands for "truncated") if $\lambda \geqq 1$ and $x \in S$ imply $\lambda x \in S$. Given a set S in H, we put

$$S^0 = \{y \in H; (x, y) \geqq 0 \quad \text{for} \quad \text{all} \quad x \in S\},$$

$$S^\Delta = \{y \in H; (x, y) \geqq 1 \quad \text{for} \quad \text{all} \quad x \in S\}.$$

Then the following properties are easily verified:

(1.1) S^0 is a non-empty closed convex cone containing 0; S^Δ is a closed convex T-cone.

(1.2) $S_1 \subset S_2$ implies $S_1^0 \supset S_2^0$ and $S_1^\Delta \supset S_2^\Delta$.

(1.3) $S^0 = H$ if and only if either $S = \emptyset$ or $S = \{0\}$; $S^\Delta = H$ if and only if $S = \emptyset$.

(1.4) $\bar{S}^0 = S^0$ and $\bar{S}^\Delta = S^\Delta$, where $\bar{S}$ denotes the closure of S in H.

(1.5) If S is closed convex, then $S^\Delta = \emptyset$ if and only if $S \ni 0$.

LEMMA 1.1. (a) *If* $S \neq \emptyset$, *then* S^{00} *is the smallest closed convex cone containing* S.

(b) *If* $S^{\Delta\Delta} \not\ni 0$ (*equivalently* $S^\Delta \neq \emptyset$), *then* $S^{\Delta\Delta}$ *is the smallest closed convex*