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Extensions of Group Actions

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§1. Introduction

Let G be a finite group, A be a discrete abelian group, and

$$0 \longrightarrow A \xrightarrow[\subset]{} K \xrightarrow{j} G \longrightarrow 1$$

be an extension with the associated operator $\phi: G \to \mathrm{Aut}(A)$, $\phi(j(k))(a) = kak^{-1}$. Also let X be a connected CW-complex with a G-action $\beta: G \times X \to X$ or $\beta: G \to \mathrm{Homeo}(X)$ satisfying $\phi(\mathrm{Ker}\,\beta) = 1_A$ and $\pi: P \to X$ be a principal A-bundle with an A-action $\alpha: A \times P \to P$.

In this paper, we study the existence and the enumeration of K-actions γ on P such that the following diagram is commutative:

$$\begin{array}{ccccc} A \times P & \xrightarrow[\subset]{} & K \times P & \xrightarrow{j \times \pi} & G \times X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ P & = & P & \xrightarrow{\pi} & X. \end{array}$$

We call such a K-action γ on P an *extended K-action* on P of α over β. In [3], A. Hattori and T. Yoshida have studied this problem for $K = A \times G$, where A is a product of a torus group and a discrete abelian group, (cf. also [2, p. 23, Remark]).

By considering the G-action (2.8) on $[X, BA]$ of all equivalence classes of principal A-bundles over X, we define the map

$$\Theta: [X, BA]^G \longrightarrow H^2_\phi(G, A) \tag{2.17}$$

from the set $[X, BA]^G$ of all G-invariant classes to the cohomology $H^2_\phi(G, A)$ of the group G with coefficients in the G-module A by ϕ. Then we have the following existence theorem:

THEOREM 3.3. *A principal A-bundle P admits an extended K-action of α over β if and only if*

$$[P] \in [X, BA]^G \quad \textit{and} \quad \Theta([P]) = \omega([K]),$$

where ω: Opext$(G, A, \phi) \to H^2_\phi(G, A)$ *is the bijection given in* [4, Ch. IV, Th. 4.1].

As an application, we obtain