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# *Finiteness Conditions for Abelian Ideals and Nilpotent Ideals in Lie Algebras*

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## 1. Introduction

Let us first recall the definitions of several classes of Lie algebras over a field.

$L \in \lhd \mathfrak{A}$-Fin; every abelian ideal of $L$ is finite-dimensional.

$L \in \lhd \mathfrak{N}$-Fin; every nilpotent ideal of $L$ is finite-dimensional.

$L \in$ Max-$\lhd \mathfrak{A}$, Max-$\lhd \mathfrak{N}$, Max-$\lhd \mathrm{E}\mathfrak{A}$; $L$ satisfies the maximal condition for abelian, nilpotent and soluble ideals respectively.

$L \in$ Min-$\lhd \mathfrak{A}$, Min-$\lhd \mathfrak{N}$, Min-$\lhd \mathrm{E}\mathfrak{A}$; $L$ satisfies the minimal condition for abelian, nilpotent and soluble ideals respectively.

For the above classes, R. K. Amayo and I. Stewart have asked the following among "Some open questions" at the end of their book [1]:

*Question* 23. *Is* $\lhd \mathfrak{N}$-Fin *equal to* $\lhd \mathfrak{A}$-Fin?

*Question* 24. *Are there any inclusions between* Max-$\lhd \mathfrak{A}$, Max-$\lhd \mathfrak{N}$, Max-$\lhd \mathrm{E}\mathfrak{A}$; Min-$\lhd \mathfrak{A}$, Min-$\lhd \mathfrak{N}$, Min-$\lhd \mathrm{E}\mathfrak{A}$?

The purpose of this paper is to give the negative answer to Question 23 and a partial answer to Question 24.

## 2. Tensorial extensions

Let $\mathfrak{k}$ be an arbitrary field. If $A$ is a commutative associative algebra over $\mathfrak{k}$ and $S$ is a Lie algebra over $\mathfrak{k}$, we can as in [2] define the Lie algebra $A \otimes_{\mathfrak{k}} S$ over $\mathfrak{k}$ with multiplication

$$[a \otimes s, a' \otimes s'] = (aa') \otimes [s, s'] \qquad (a, a' \in A; s, s' \in S).$$

If $S$ is a Lie algebra over $\mathfrak{k}$, for any $s \in S$ we define the adjoint transformation $s^*$ of $S$ by $xs^* = [x, s]$ $(x \in S)$. As $s$ varies throughout $S$, these transformations generate a subalgebra $\mathfrak{M}$ of the associative algebra $End_{\mathfrak{k}}(S)$ of all linear transformations of $S$. Now $S$ is a right $\mathfrak{M}$-module with the above action. The centroid $\mathfrak{C}$ of $\mathfrak{M}$ is the algebra of all $\mathfrak{M}$-endomorphisms of $S$, these being regarded as left operaters on $S$. The Lie algebra $S$ is central simple if it is simple and the centroid $\mathfrak{C}$ is the ground field $\mathfrak{k}$ with its standard action on $S$. I. Stewart has shown in [2] that if $S$ is central simple and $A$ has an identity then every ideal of $A \otimes_{\mathfrak{k}} S$ is of the form $U \otimes S$ where $U$ is an ideal of $A$. From this result, we have