HIROSHIMA MATH. J.
8 (1978), 269–299

Differential Calculus in Linear Ranked Spaces

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(Received December 26, 1977)

§0. Introduction

The theory of differential calculus in Banach spaces has been already established (cf. e.g., J. Dieudonné [1, Ch. VIII]), and there have been various attempts to construct differential calculus in more general linear spaces. For example, A. Fröhlicher and W. Bucher [2] have studied in linear spaces with limit structures based on filters, H. H. Keller [3] has studied the notion of C^p-mappings in locally convex spaces, and S. Yamamuro [7] has introduced the notion of equicontinuous differentiability in topological linear spaces.

In this paper, we try to develop differential calculus in linear ranked spaces. The notion of ranked spaces was first introduced by K. Kunugi [4]; and M. Yamaguchi [6] considered differentiation in linear ranked spaces. Using a modified formulation of linear ranked spaces given in M. Washihara [5, II], we shall study differentiation further than [6] and show that many important results in differential calculus can be included in our theory. In many respects, our construction of the theory and the methods of proofs are analogous to those in [2] and [7], though the underlying structures of the spaces are different.

We prepare in §1 some notions and results on linear ranked spaces. We define the notion of R-differentiability in §2, and prove the chain rule (Theorem 2.2) and the mean value theorem (Theorem 3.1). Further we study the Gâteaux differentiability in §4, and the invertibility of differentiable mappings in §5 (Theorems 5.2–5). Finally in §6, the higher derivatives are considered.

The author would like to thank Professors M. Sugawara and F-Y. Maeda for their many helpful comments in reading the whole manuscripts.

§1. Linear ranked spaces

Let E be a linear space over the real field $\boldsymbol{R}$. Suppose that a sequence $\{\mathfrak{V}_n\}_{n=0}^{\infty}$ of families of subsets in E is given to satisfy the following condition (E.1):

(E.1) *$0 \in V$ for any $V \in \mathfrak{V} = \cup_{n=0}^{\infty} \mathfrak{V}_n$, $E \in \mathfrak{V}_0$; and for any $V \in \mathfrak{V}$ and for any integer $n \geqq 0$, there are another integer $m > n$ and $U \in \mathfrak{V}_m$ such that $U \subset V$.*

Sets in $\mathfrak{V}_n$ are called *preneighborhoods of the origin* 0 *with rank n.*