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Fine Differentiability of Riesz Potentials

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1. Introduction

In the n-dimensional Euclidean space R^n, we are concerned with the differentiability properties of Riesz potential U_α^μ of order α, $0<\alpha<n$, of a non-negative measure μ. The potential U_α^μ may fail to be differentiable at any point of R^n, since U_α^μ may take the value ∞ on a countable dense subset of R^n. We are therefore motivated to relax the requirement in the definition of differentiability; in fact, if we restrict the set of approach to x^0, then we may be able to conclude

$$\lim_{x\to x^0, x\notin E} \frac{|U_\alpha^\mu(x) - U_\alpha^\mu(x^0) - L(x-x^0)|}{|x-x^0|} = 0,$$

where $L=L_{x^0}$ is a linear function. The following problems are proposed here:

(i) Characterize the excluded set E in an appropriate manner.

(ii) Evaluate the size of the set of all x^0 at which U_α^μ is not differentiable in such a sense.

Before finding answers to these problems, we fix some notation which will be used in this note. For a point $x=(x_1,\dots, x_n)\in R^n$ and a multi-index $\gamma=(\gamma_1,\dots, \gamma_n)$, we define

$$x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}, \quad (\partial/\partial x)^\gamma = (\partial/\partial x_1)^{\gamma_1}\cdots(\partial/\partial x_n)^{\gamma_n},$$

$$\gamma! = \gamma_1!\cdots\gamma_n!, \quad |\gamma| = \gamma_1 + \cdots + \gamma_n.$$

We denote by R_α the Riesz kernel of order α. Fix a point $x^0\in R^n$ and set

$$K_m(x, y) = R_\alpha(x-y) - \sum_{|\gamma|\leqq m} \frac{1}{\gamma!}(x-x^0)^\gamma \frac{\partial^\gamma R_\alpha}{\partial x^\gamma}(x^0-y)$$

for a positive integer m.

A set E is said to be α-thin at x^0 either if $x^0\notin\overline{E\backslash\{x^0\}}$ (the closure of $E\backslash\{x^0\}$) or if $x^0\in\overline{E\backslash\{x^0\}}$ and there is a non-negative measure μ satisfying

$$\liminf_{x\to x^0, x\in E\backslash\{x^0\}} U_\alpha^\mu(x) > U_\alpha^\mu(x^0).$$

Our first aim is to prove the following theorem.