Hiroshima Math. J.
8 (1978), 469–489

KO-Groups of Lens Spaces Modulo Powers of Two

Dedicated to Professor A. Komatu on his 70th birthday

Kensô Fujii, Teiichi Kobayashi, Katsumi Shimomura
and Masahiro Sugawara

(Received March 13, 1978)

§1. Introduction

The K- and KO-rings of the standard lens space $L^n(m)=S^{2n+1}/Z_m$ mod m are investigated by several authors, and the structures of the reduced K- and KO-rings $\tilde{K}(L^n(m))$ and $\widetilde{KO}(L^n(m))$ are determined by J. F. Adams [1, Th.7.3–4] when $m=2$ ($L^n(2)=RP^{2n+1}$ is the real projective space), and T. Kambe [3] when m is an odd prime. Furthermore, the additive groups $\tilde{K}(L^n(p^r))$ (p: prime) and $\widetilde{KO}(L^n(p^r))$ (p: odd prime) are determined by N. Mahammed [9, Th.3], and an explicit additive base of $\tilde{K}(L^n(p^r))$ (p: odd prime) is given in [5, Th.1.7].

In this note, we shall determine the additive structure of

$$\widetilde{KO}(L^n(2^r)) \qquad \text{for any} \quad r \geqq 2.$$

Let ρ *be the non-trivial real line bundle over* $L^n(2^r)$, and η *be the canonical complex line bundle over* $L^n(2^r)$, i.e., the induced bundle of the canonical complex line bundle over the complex projective space CP^n by the natural projection $\pi: L^n(2^r) \to CP^n$. Then we can prove the following

Proposition 1.1. *The reduced KO-ring* $\widetilde{KO}(L^n(2^r))$ $(r \geqq 2)$ *is generated by the stable classes*

$$\kappa = \rho - 1, \quad \bar{\sigma} = r\eta - 2 \ (r\eta \text{ is the real restriction of } \eta)\,; \tag{1.2}$$

and there hold the following relations:

$$\bar{\sigma}^i = 0 \qquad \text{for} \quad i > [n/2] + \varepsilon, \ \varepsilon = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 4, \\ 0 & \text{otherwise}; \end{cases} \tag{1.3}$$

$$\bar{\sigma}(r-1) = 2\kappa, \quad \kappa^2 = -2\kappa, \tag{1.4}$$

$$\kappa\bar{\sigma} = -2\kappa + \textstyle\sum_{s=1}^{r-2} \{(2+\bar{\sigma})\bar{\sigma}(s) \prod_{t=s+1}^{r-2} (2+\bar{\sigma}(t))\}, \tag{1.5}$$

where $\bar{\sigma}(s) = \bar{\sigma}^{2^s} + \sum_{j=1}^{2^s-1} y_{sj}\bar{\sigma}^j \in \widetilde{KO}(L^n(2^r))$ *is given inductively by*

$$\bar{\sigma}(0) = \bar{\sigma}, \quad \bar{\sigma}(s) = 4\bar{\sigma}(s-1) + \bar{\sigma}(s-1)^2 \qquad (0 < s < r). \tag{1.6}$$