HIROSHIMA MATH. J.
10 (1980), 153–161

A note on noetherian Hilbert rings

Kazunori FUJITA and Shiroh ITOH

(Received August 23, 1979)

Introduction. All rings considered here are commutative with identity. In this note, we give two examples of noetherian Hilbert rings. The most famous example of a noetherian Hilbert domain is an affine domain over a field. Such an integral domain is equicodimensional, i.e. its all maximal ideals have the same height. Noetherian Hilbert domains with maximal ideals of different height are given in [1], [5], [6], [10] and [11]. Krull's example in [6] is obtained by a localization of $K[X, Y]$, where K is a countable, algebraically closed field. Heinzer in [5] constructs a noetherian Hilbert domain with maximal ideals of preassigned height, and subsequently in [1] and [10] the same examples as Heinzer's are constructed by making use of the following proposition in [4, (10. 5. 8)]: Let A be a noetherian ring and let s be a non-nilpotent element contained in $\mathrm{rad}(A)$. Then A_s is a Hilbert ring.

By the way, in [6] and [11], two dimensional noetherian Hilbert domains with only a finite number of height one maximal ideals are constructed. However almost all noetherian Hilbert domains already known have the following property: Let $\mathfrak{M}$ be a maximal ideal of a noetherian Hilbert domain A. Then, if $n = ht(\mathfrak{M}) \geq 2$, A has infinitely many height n maximal ideals.

In Section 1, we show that if A is a noetherian ring containing an uncountable field and if S is a multiplicative subset of A generated by countably many elements of $\mathrm{rad}(A)$, then $S^{-1}A$ is a Hilbert ring. In Section 2, we construct a noetherian Hilbert domain with a preassigned number of maximal ideals of preassinged height by making use of a modification of Krull's method in [6, p. 371].

Notation. Let A be a ring. Then

$\mathrm{Max}(A) = \{\mathfrak{P} \in \mathrm{Spec}(A);\ \mathfrak{P}$ is a maximal ideal in $A\}$,

$Ht_n(A) = \{\mathfrak{P} \in \mathrm{Spec}(A);\ ht(\mathfrak{P}) = n\}$,

$\mathrm{rad}(A) = \bigcap_{\mathfrak{P} \in \mathrm{Max}(A)} \mathfrak{P}$.

Let $\mathfrak{p}$ be a prime ideal in a ring A. Then

$U(\mathfrak{p}) = \{\mathfrak{P} \in \mathrm{Spec}(A);\ \mathfrak{P} \supset \mathfrak{p}$ and $ht(\mathfrak{P}/\mathfrak{p}) = 1\}$.

$\mathbf{C}$ = the field of complex numbers.

$\mathbf{N}$ = the set of natural numbers.

1. We need some preliminary results.

LEMMA 1. *Let A be a noetherian ring. Then A is a Hilbert ring if and*