HIROSHIMA MATH. J.
10 (1980), 11–53

A finite-difference method on a Riemannian manifold

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(Received April 24, 1979)

Introduction

The aim of the present paper is to extend the results in the paper "A finite-difference method on a Riemann surface" [10] to the higher dimensional case.

In Chapter I, we establish orthogonal decomposition theorems concerning difference forms on an n-dimensional polyhedron ($2 \leqq n < \infty$) which give an analogue to de Rham-Kodaira's theory on a Riemannian manifold (cf. Kodaira [8] and de Rham [14]). Here our definition of a polyhedron differs from the ordinary one based on a triangulation and is based on a polyangulation of an n-dimensional manifold (see §**1.1**). A p-difference (p-th order difference form; $0 \leqq p \leqq n$) on a polyhedron is defined as a function on a p-chain which takes a complex value at each oriented p-simplex (see §**2.1**). In order to set the definition of a conjugate difference form which answers our purpose, we introduce the concepts of a conjugate polyhedron and of a complex polyhedron (see §**1.3**). A theory of harmonic difference forms on the complex polyhedron which is analogous to the theory of differential forms on a Riemannian manifold, is then established (cf. Mizumoto [10] and [11] in the 2- and 3-dimensional cases). Eckmann [6] treated a boundary value problem of a harmonic difference form on a polyhedron (Komplex). Our method, which makes an effective use of a conjugate difference on a conjugate polyhedron, is different from his.

In Chapter II, we shall concern ourselves with the problem of approximating a harmonic p-th order differential form on a Riemannian manifold by harmonic p-th order difference forms. We define a sequence $\{\boldsymbol{K}_i\}_{i=0}^{\infty}$ of normal subdivisions of a normal complex polyhedron $\boldsymbol{K}_0$ (see §**1.6**) and a Riemannian manifold M based on $\boldsymbol{K}_0$ (see §**1.7**). Then we shall discuss the norm convergence of smooth extensions of harmonic difference forms on $\boldsymbol{K}_i$, $i=0, 1, 2, \ldots$, to a harmonic differential on M (see Theorems 5.1, 5.2, 5.3 and 5.4, and cf. §**5.2** for the definition of smooth extension). In our present method, the harmonicity of the limit differential form of smooth extensions of harmonic difference forms and that of their conjugate difference forms are simultaneously shown. Our method is based on the fact that the smooth extensions of a harmonic difference form and its conjugate difference form are closed differential forms, so that their limit differential forms in the Hilbert space of differential forms are a pair of closed and conjugate closed ones, and thus a pair of harmonic and conjugate harmonic ones.