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Ideally finite Lie algebras

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Introduction

Recently Stitzinger [4] presented several equivalent conditions for a subalgebra to be an ω-step ascendant subalgebra in a locally solvable, ideally finite Lie algebra. On the other hand, Tôgô [5] introduced the concept of weakly ascendant subalgebras generalizing that of ascendant subalgebras and Kawamoto [2] considered E_∞-pairs of subalgebras to study ascendancy in Lie algebras.

The purpose of this paper is first to generalize and sharpen the results of Stitzinger [4] by using the concepts of weakly ascendant subalgebras, E_∞-pairs of subalgebras and others, and secondly to characterize the class of locally solvable, ideally finite Lie algebras and similar classes.

Section 2 is devoted to searching several equivalent conditions for a subalgebra to be a weakly ascendant subalgebra in a certain Lie algebra (Theorems 2.1, 2.2, 2.3 and Corollary 2.4). In Section 3 we shall show that if L is a locally solvable, ideally finite Lie algebra and H is a subalgebra of L, then the condition $H \lhd^\omega L$ is equivalent to each of the following: (a) $H \operatorname{asc} L$; (b) $H \leq^\omega L$; (c) $H \operatorname{wasc} L$; (d) (H, L) is an E-pair; (e) (H, L) is an E_∞-pair; (f) $L = H + L_0(h)$ for all $h \in H$; (g) $L_1(h) \subseteq H$ for all $h \in H$; (h) $H \lhd^\omega K$ for any subalgebra K of L containing H; (i) $H \lhd^\omega \langle H, x \rangle$ for any $x \in L$; (j) $H \lhd^\omega \langle H, [x, H] \rangle$ for any $x \in L$; (k) For any $x \in L$, there exists an $n = n(x)$ such that $H \lhd^\omega \langle H, [x, {}_nH] \rangle$ (Theorems 3.1 and 3.2). This sharpens [4, Theorems 1 and 3]. We shall also give a simple proof of [4, Theorem 2] in Proposition 3.3.

In Section 4 we shall show that all ideally finite Lie algebras belonging to a class $\mathfrak{X}$ are precisely locally solvable, ideally finite Lie algebras, if $\mathfrak{X}$ is a class of Lie algebras being between the class $\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A}$ and the class $\mathfrak{X}_0$ of all Lie algebras in which every non-zero finite-dimensional subalgebra is non-perfect (Theorem 4.3). Finally in Section 5 we shall show that (a) the class of locally nilpotent, subideally finite Lie algebras coincides with the class of Baer algebras and (b) the class of locally nilpotent, ascendantly finite Lie algebras is contained in the class of Gruenberg algebras (Theorem 5.3). We shall also give two examples showing that there are no inclusions between the class of locally solvable, ideally finite Lie algebras and the class of locally nilpotent, subideally finite Lie algebras (Example 5.7).