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Weakly serial subalgebras of Lie algebras

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Introduction

Recently Stitzinger [7] presented some equivalent conditions for a subalgebra to be an ω-step ascendant subalgebra in a locally solvable, ideally finite Lie algebra. Subsequently Tôgô, Honda and Sakamoto [9] generalized and sharpened the results of [7] by using the concepts of weakly ascendant subalgebras, E-pairs and E_{∞}-pairs of subalgebras. On the other hand, Stewart [6] investigated properties of serial subalgebras of a locally finite Lie algebra.

In this paper we shall introduce the concept of weakly serial subalgebras of a Lie algebra generalizing that of serial subalgebras. The purpose of this paper is first to investigate properties of weakly serial subalgebras of a locally finite Lie algebra, and secondly to generalize the results of [6] by using the concept of weakly serial subalgebras, and thirdly to develop the results analogous to those of [9, §§2 and 3] by using the concepts of weakly serial subalgebras and weakly descendant subalgebras.

In Section 2 we shall show that in a locally solvable, locally finite Lie algebra all the weakly serial subalgebras are precisely the serial subalgebras (Theorem 2.7). We shall also show that if H is a subalgebra of a locally finite Lie algebra L, then the condition $H \operatorname{wser} L$ is equivalent to each of the following conditions: (a) $H \operatorname{wser} \langle H, X\rangle$ for any finite subset X of L; (b) $H \operatorname{wser} \langle H, x\rangle$ for any $x \in L$; (c) $H \operatorname{wser} \langle H, [x,_n H]\rangle$ for any $x \in L$ $(n \in \mathbf{N})$; (d) For any $x \in L$ there exists an $n = n(x) \in \mathbf{N}$ such that $H \operatorname{wser} \langle H, [x,_n H]\rangle$ (Theorem 2.8). Furthermore, we shall show that for a subalgebra H of a locally finite Lie algebra L, $H \operatorname{wser} L$ if and only if $\lambda_{L\mathfrak{N}}(H) \lhd L$ and $H/\lambda_{L\mathfrak{N}}(H) \subseteq \mathfrak{e}(L/\lambda_{L\mathfrak{N}}(H))$ (Theorem 2.12). This generalizes [6, Theorem 5]. In Section 3 we shall generalize [9, Theorems 2.1 and 2.2] (Theorem 3.1). We shall also show that if H is a subalgebra of a locally solvable, ideally finite Lie algebra L, then the condition $H \lhd^{\omega} L$ is equivalent to each of the following conditions: (a) $H \operatorname{ser} L$; (b) $H \operatorname{wser} L$ (Theorem 3.3). In Section 4 we shall show that if L is an abelian-by-nilpotent Lie algebra and if σ is an infinite ordinal, then all the σ-step weakly descendant subalgebras of L are precisely the σ-step descendant subalgebras of L (Corollary 4.3). We shall also show that if L is an ideally finite Lie algebra such that $L/\zeta_1(L)$ is countable-dimensional and if H is a weakly serial nilpotent subalgebra of L, then H is an $\omega 2$-step weakly descendant subalgebra of L (Theorem 4.5 and Corollary 4.6). In Section 5 we