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Growth estimates for the coefficients of generalized formal solutions, and representation of solutions using Laplace integrals and factorial series

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0. Introduction

a) Functions represented by Laplace integrals:

A question of considerable interest in the theory of meromorphic differential equations as well as in other fields of analysis is, given a function $f(z)$ that has an asymptotic expansion as $z \to \infty$ (in some sector), can $f(z)$ be represented by means of a (generalized) Laplace integral such that one can find a convergent expansion of the integrand (in some neighborhood of zero) in terms of the asymptotic expansion of $f(z)$?

In 1918, F. Nevanlinna [17] has given an answer to this question, generalizing some earlier results of G. N. Watson [23]: Suppose that (for fixed reals $a \geq 0$ and $d > 0$) the function $f(z)$ is analytic in the sector

$$S = \{|z| > a, \quad |\arg z| < \pi/(2d)\}$$

(note that throughout this paper the variable z is on the Riemann surface of the Logarithm, hence in case $d < 1/2$ the function $f(z)$ may be multi-valued). Furthermore, assume the existence of a formal power series

$$\textstyle\sum_1^\infty f_k z^{-k}$$

such that for some positive constant K and every sufficiently large integer j

$$|z^j(f(z) - \textstyle\sum_1^{j-1} f_k z^{-k})| \leq K^j \Gamma(1+j/d) \qquad (z \in S). \tag{0.1}$$

Then it is easy to conclude

$$|f_j| \leq K^j \Gamma(1+j/d) \quad \text{for sufficiently large} \quad j,$$

hence the power series

$$\psi(u) = \textstyle\sum_1^\infty f_k u^k / \Gamma(1+k/d) \tag{0.2}$$

converges for $|u| < K^{-1}$. Representing $\psi(u)$ as a generalized inverse Laplace integral over $f(z)$, F. Nevanlinna proved that $\psi(u)$ can be analytically continued into an (explicitly given) region containing the positive real axis, and for every