Hiroshima Math. J.
12 (1982), 421–433

Fourier-like transformation and a representation of the Lie algebra $\mathfrak{so}(n+1, 2)$

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(Received January 14, 1982)

1. Introduction

The space M of non-zero cotangent vectors to the unit sphere S^n is an $SO(n+1, 2)$-homogeneous symplectic manifold. The geometry of the $SO(n+1, 2)$-action is studied by several authors. (See Akyildiz [1], Onofri [10], [11], Rawnsley [14], Souriau [19] and Wolf [24], [25].) The present note is motivated by Wolf [24], [25]. We consider the problem of "quantizing" this $SO(n+1, 2)$-action. The standard procedure of geometric quantization does not work because there are no $SO(n+1, 2)$-invariant polarizations. (See Elhadad [2], Ozeki and Wakimoto [12], Wakimoto [22] and Wolf [24].) We will work in the framework of Lie algebras rather than groups. The Lie algebra $\mathfrak{so}(n+1, 2)$ is realized as a Poisson subalgebra $\mathfrak{G}$. By integration of the Hamiltonian vector fields associated with elements of $\mathfrak{G}$, we get the symplectic action of $SO(n+1, 2)$ on M. To construct a representation of $\mathfrak{so}(n+1, 2)$, we use a pair of transversal polarizations: one is the vertical polarization Q and the other is a partially complex polarization P invariant under the geodesic flow. The space $\Gamma_Q(\boldsymbol{L}\otimes L^Q)$ of smooth Q-horizontal sections of a complex line bundle $\boldsymbol{L}\otimes L^Q$ over M is naturally identified with $C^\infty(S^n)$. While there exist no smooth P-horizontal sections in $\Gamma(\boldsymbol{L}\otimes L^P)$ except for zero-section, so we must consider "singular" sections. The supports of singular P-horizontal sections are in a disjoint union of hypersurfaces M_m $(m=0, 1, 2,\dots)$ in M. Each M_m is identified with the Stiefel manifold $SO(n+1)/SO(n-1)$, which is an $SO(2)$-principal bundle over the Grassmann manifold $SO(n+1)/(SO(2)\times SO(n-1))$. The Grassmann manifold is an $SO(n+1)$-homogeneous complex manifold. Let L_m be the $SO(n+1, \mathbf{C})$-homogeneous holomorphic line bundle over the Grassmann manifold given in Kowata and Okamoto [8]. Holomorphic sections of L_m are identified with functions on $SO(n+1)/SO(n-1)$. If we identify M_m with this Stiefel manifold, then holomorphic sections of L_m are identified with functions on M_m. Since $\boldsymbol{L}\otimes L^P$ is a trivial bundle over M, these functions are identified with singular sections of $\boldsymbol{L}\otimes L^P$ with supports in M_m. These sections are P-horizontal. The correspondence: a holomorphic section of $L_m \mapsto$ a P-horizontal section of $\boldsymbol{L}\otimes L^P$ with support in M_m, is bijective. Thus, the consideration of the P-horizontal sections is equivalent to that of all the holomorphic sections of L_m $(m=0, 1, 2,\dots)$