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Joins of weak subideals of Lie algebras

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Introduction

Maruo [4] introduced the concept of weak ideals of Lie algebras generalizing that of subideals and investigated pseudo-coalescency of classes of Lie algebras. Tôgô [6] introduced the concept of weakly ascendant subalgebras of Lie algebras generalizing those of weak ideals and ascendant subalgebras. Weak ideals are called weak subideals in [6]. A class $\mathfrak{X}$ of Lie algebras is pseudo-coalescent [4] if in any Lie algebra the join of any pair of a subideal and a weak subideal belonging to $\mathfrak{X}$ is always a weak subideal belonging to $\mathfrak{X}$. However, it might be meaningless to consider classes $\mathfrak{X}$ such that in any Lie algebra the join of any pair of weak subideals belonging to $\mathfrak{X}$ is always a weak subideal belonging to $\mathfrak{X}$, for there exists a Lie algebra L such that the join of some pair of 1-dimensional weak subideals of L is not a weak subideal of L and is non-abelian simple (cf. [3, Example 5.1]). In this paper we shall investigate several classes of Lie algebras in which the join of any pair of weak subideals (resp. subideals) is always a weak subideal (resp. a subideal).

In Section 2 we shall show that if a Lie algebra L belongs to one of the classes $\overline{\mathfrak{D}}\mathfrak{A}$, $\mathfrak{N}\overline{\mathfrak{M}}_1$ and $\mathfrak{N}\mathfrak{A}_1$ (resp. the classes $\mathfrak{D}\mathfrak{A}$, $\mathfrak{N}\mathfrak{M}_1$ and $\mathfrak{N}\mathfrak{A}_1$), then the set $\mathscr{S}_L(\mathrm{wsi})$ (resp. the set $\mathscr{S}_L(\mathrm{si})$) of all weak subideals (resp. all subideals) of L is a sublattice of the lattice $\mathscr{S}_L(\leq)$ of all subalgebras of L (Theorem 2.11). In Section 3 we shall show that if a Lie algebra L belongs to one of the classes $\mathfrak{A}_1$, $\mathfrak{F}\cap(\mathfrak{N}\overline{\mathfrak{M}}_1)$ and $\mathfrak{F}\cap(\mathfrak{N}\mathfrak{A}_1)$ (resp. the classes $\mathfrak{A}_1$, $\mathfrak{F}\cap(\mathfrak{N}\mathfrak{M}_1)$ and $\mathfrak{F}\cap(\mathfrak{N}\mathfrak{A}_1)$), then $\mathscr{S}_L(\mathrm{wsi})$ (resp. $\mathscr{S}_L(\mathrm{si})$) is a complete sublattice of $\mathscr{S}_L(\leq)$ (Theorem 3.5). In Section 4 we shall construct Lie algebras L such that $\mathscr{S}_L(\mathrm{wsi})$ is a sublattice of $\mathscr{S}_L(\leq)$ but is not a complete sublattice of $\mathscr{S}_L(\leq)$ (Examples 4.1 and 4.2). We shall also construct a Lie algebra L such that $\mathscr{S}_L(\mathrm{wsi})$ is a complete sublattice of $\mathscr{S}_L(\leq)$ (Example 4.3).

Here $\mathfrak{A}$ (resp. $\mathfrak{N}$, $\mathfrak{F}$) is the class of abelian (resp. nilpotent, finite-dimensional) Lie algebras, $\overline{\mathfrak{D}}$ (resp. $\mathfrak{D}$) is the class of Lie algebras in which every subalgebra is a weak subideal (resp. a subideal), and $\overline{\mathfrak{M}}_1$ (resp. $\mathfrak{M}_1$) is the class of Lie algebras in which every weak subideal (resp. every subideal) is an ideal; and $\mathfrak{A}_1$ is the class consisting of either abelian Lie algebras or metabelian Lie algebras L with $\dim(L/L^2)=1$. For classes $\mathfrak{X}$, $\mathfrak{Y}$ of Lie algebras, $\mathfrak{X}\mathfrak{Y}$ is the class of Lie algebras L having an $\mathfrak{X}$-ideal I such that $L/I\in\mathfrak{Y}$.