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Weak boundary components in R^N

Dedicated to Professor M. Ohtsuka for his 60th birthday

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Introduction

Let D be a bounded plane domain and γ be a component of the boundary of D consisting of a single point. It is called by Sario [7] weak if its image under any conformal mapping of D consists of a single point. Jurchescu [3] gave a characterization of the weakness by means of extremal length.

In the N-dimensional euclidean space R^N ($N \geqq 3$), Sario [8] introduced the notion of the capacity c_γ of a subboundary γ of a domain in R^N and posed the following question: Is a component γ of a compact set E in R^N a point if and only if $c_\gamma = 0$ for the domain $R^N - E$ ([8, p. 110])? A boundary component γ is called weak if $c_\gamma = 0$.

In the present paper we shall be concerned with this question. Let D be a domain in R^N and E be a compact set such that $\gamma = \partial E$ is a subboundary of D. We shall give an example (Example 1) in which γ is a point but $c_\gamma \neq 0$. Moreover, in case γ is an isolated subboundary, we shall show (Theorem 2) that $c_\gamma = 0$ if and only if the Newtonian capacity $C_2(E) = 0$. Since there exists a continuum E with $C_2(E) = 0$ (cf. [1]), it follows that even for a continuum E, $\gamma = \partial E$ can be weak.

In §4, we shall give a characterization of the weakness by means of the extremal length of order 2. Let B be a ball in D and $\hat{\Gamma}$ denote the family of curves in the Kerékjártó-Stoïlow compactification each of which connects γ and B. We shall show (Theorem 4) that $c_\gamma = 0$ if and only if the extremal length $\lambda_2(\hat{\Gamma}) = \infty$. In §5, we shall derive the modular criterion of the weakness which is well known for Riemann surfaces (cf. [9]).

§ 1. Preliminaries

Let R^N ($N \geqq 3$) be the N-dimensional euclidean space. We shall denote by $x = (x_1, x_2, \dots, x_N)$ a point in R^N, and set $|x| = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$. For a set E in R^N, we denote by ∂E and $\bar{E}$ the boundary and the closure of E with respect to the N-dimensional Möbius space $R^N \cup \{\infty\}$, respectively. Let $B(r, x)$ denote the open N-ball of radius r and centered at x. The area of $\partial B(1, x)$ will be written as ω_N. For a function u defined in a domain G, we let ∇u denote the gradient of