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The Bergman kernel function for symmetric Siegel domains of type III

Dedicated to Professor K. Murata for his 60th birthday

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It is known (Wolf-Korányi [7]) that every hermitian symmetric space of noncompact type has a standard realization as a Siegel domain of type III. In this note we give an explicit formula for the Bergman kernel function of such a symmetric Siegel domain.

The general definition of Siegel domain of type III was given by Pyatetskii-Shapiro [4] as follows. Let U, V and W be complex vector spaces. Let $U_{\boldsymbol{R}}$ be a real form of U, Ω an open convex cone in $U_{\boldsymbol{R}}$, and B a bounded domain in W. Given any $w \in B$, let Φ_w be a semi-hermitian form of $V \times V$ to U, i.e., $\Phi_w = \Phi_w^h + \Phi_w^b$ where Φ_w^h is hermitian relative to the complex conjugation of U over $U_{\boldsymbol{R}}$ and Φ_w^b is symmetric $\boldsymbol{C}$-bilinear. Then the domain

$$\{(u, v, w) \in U \oplus V \oplus W;\ \mathrm{Im}\, u - \mathrm{Re}\, \Phi_w(v, v) \in \Omega,\ w \in B\}$$

is called a Siegel domain of type III. Siegel domains of type II are degenerate special case $W=0$, i.e., $B=(0)$, $\Phi_0^b=0$ and Φ_0^h is positive definite relative to Ω.

For Siegel domains of type II (not necessarily symmetric nor homogeneous), an explicit formula for the Bergman kernel was given by Gindikin [1, Theorem 5.4] in terms of a certain integral over the dual cone of Ω (see also Korányi [3, Proposition 5.3]).

Every hermitian symmetric space of noncompact type can be written as G/K, where G is a connected semi-simple linear Lie group and K is a maximal compact subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. We denote the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ by $\mathfrak{g}_{\boldsymbol{C}}$, $\mathfrak{k}_{\boldsymbol{C}}$, $\mathfrak{p}_{\boldsymbol{C}}$, respectively. Then $\mathfrak{p}_{\boldsymbol{C}}$ is decomposed into the direct sum of two complex subalgebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$, which are $(\pm i)$-eigenspaces of the complex structure of $\mathfrak{p}$, respectively, and are abelian subalegbras of $\mathfrak{g}_{\boldsymbol{C}}$ normalized by $\mathfrak{k}_{\boldsymbol{C}}$.

Let $G_{\boldsymbol{C}}$ be the complexification of G and let $P^{\pm}$, $K_{\boldsymbol{C}}$ be the connected subgroups of $G_{\boldsymbol{C}}$ corresponding to $\mathfrak{p}^{\pm}$, $\mathfrak{k}_{\boldsymbol{C}}$, respectively. It is known that the map $\mathfrak{p}^+ \times K_{\boldsymbol{C}} \times \mathfrak{p}^- \to G_{\boldsymbol{C}}$, given by $(X^+, k, X^-) \to \exp X^+ \cdot k \cdot \exp X^-$, is a holomorphic diffeomorphism onto a dense open subset $P^+ K_{\boldsymbol{C}} P^-$ of $G_{\boldsymbol{C}}$, which contains G. Therefore, every element $g \in P^+ K_{\boldsymbol{C}} P^- \subset G_{\boldsymbol{C}}$ can be written in a unique way as