Half-spherical means and boundary behaviour of subharmonic functions in half-spaces

Stephen J. Gardiner

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1. Introduction and main results

Naïm [9], Chapitre IV, has shown how the minimal fine topology and Martin boundary may be used to obtain elegant generalizations of the maximum principle and the solution to the Dirichlet problem. The purpose of this paper is to show that remarkably similar results can be obtained for half-spherical means in the Euclidean half-space $\Omega = R^{n-1} \times (0, +\infty)$.

If E is a subset of R^n, then its closure and boundary will be denoted by $\bar{E}$ and ∂E respectively. We represent points of R^n by X, Y or P, and use O for the origin; sometimes it will be convenient to write $X = (X', x_n)$, where $X' \in R^{n-1}$. The open ball of radius r centred at P will be abbreviated to $B(P, r)$ and, using σ to represent surface area measure, we write c_n for $\sigma(\partial B(O, 1))$. Another important constant is

$$\gamma_2 = (2\pi)^{-1}, \quad \gamma_n = \{(n-2)c_n\}^{-1} \quad (n \geq 3).$$

We adjoin the isolated point ∞ to the usual topology on $\bar{\Omega}$ and write $\bar{\Omega}^*$ for $\bar{\Omega} \cup \{\infty\}$. The set $\partial\Omega \cup \{\infty\}$ with the topology induced on it by $\bar{\Omega}^*$ will be abbreviated to $\partial^*\Omega$. Provided the integrals exist, we can now define the half-spherical means

$$N(f: P, r) = r^{-n-1} \int_{\partial B(P,r) \cap \Omega} x_n f(X) d\sigma(X)$$

for $P \in \partial\Omega$, and

$$N(f: \infty, r) = r^{-n} N(f: O, r^{-1}).$$

Let G denote the Green kernel for Ω. From well-known inequalities for G (see, for example, [10], Lemma 1), it follows that the function $G(X, Y)/\{x_n y_n\}$ has an extension $G^*(X, Y)$ to $\bar{\Omega} \times \bar{\Omega}$ which is jointly continuous (in the extended sense at points of the diagonal), and that

$$\gamma_n G^*(X, Y) = 2|X - Y|^{-n}/c_n \quad (Y \in \partial\Omega, X \in \bar{\Omega}\backslash\{Y\}).$$

From the Riesz decomposition theorem and [7], Theorem 2.25, it is now easy