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Gel'fand integrals and generalized derivatives of vector measures

Dedicated to Professor Takizo Minagawa on his 70th birthday

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Introduction

Let X be a real Banach space and (S, Σ, μ) a finite nonnegative complete measure space. For a general Banach space X, a μ-continuous vector measure $\nu: \Sigma \to X$ of finite variation need not be the indefinite Bochner integral of its derivative unless the values are suitably chosen. The purpose of this paper is to introduce a notion of generalized derivative which can be defined for any vector measure that is μ-continuous and of finite variation and investigate basic properties of the generalized derivatives.

The fundamental theorem of calculus for vector measures (called commonly the Radon-Nikodým theorem) need not be valid for all types of integrals. It is a notable pathology that the Bochner integral does not mimic the Lebesgue integral with regard to the fundamental theorem. In this connection various types of integrals which include the Bochner integral have been introduced through the duality theory by Birkhoff, Dunford, Gel'fand, Pettis and Phillips; and the Radon-Nikodým theorems have been established in the respective senses. Although each integral definition has its own features, the most general one among them is that of Gel'fand and the so-called Dunford-Pettis theorem is regarded as the associated fundamental theorem for vector measures with values in dual Banach spaces. Our notion of generalized derivative is also based on the Gel'fand integration theory.

Various examples of vector measures with values in nonreflexive Banach spaces such as $L^1(\mu)$ and $L^\infty(\mu)$ suggest that to an arbitrary vector measure only the differentiation in the sense of the weak*-topology of X (viewed as a subspace of its second dual unless X is a dual Banach space) may be applied. Indeed, if ν is a measure on Σ with values in a dual Banach space not possessing the Radon-Nikodým property, only the local boundedness of the set $\{\nu(E)/\mu(E): E \in \Sigma\}$ may be assumed, namely: There exists a sequence $\{S_n: n=0, 1, 2,\ldots\}$ such that $\mu(S_0)=0$, $\mu(S_n)>0$ $(n \geqq 1)$, $S=\cup_{n=0}^{\infty} S_n$ and $\{\nu(E)/\mu(E): E \in \Sigma, E \in S_n\}$ is bounded for each $n \geqq 1$. Therefore, only the relative weak*-compactness is applied to find the derivative and the derivative (supposing it is defined) is to possess only the weak*-measurability. The Dunford-Pettis theorem may apply to X-valued