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On Lie algebras with finiteness conditions

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In this paper we consider three types of questions concerning the ideal and subideal structure of Lie algebras with certain finiteness conditions. First we consider the question of finding conditions under which the join of subideals of a Lie algebra is a subideal. We prove (Theorem 1.2) that when L is a Lie algebra over a field of characteristic zero and $\{H_\lambda \mid \lambda \in \Lambda\}$ is a set of subideals of L with J their join, J is a subideal of L if and only if the set of subideals of L lying in J has a maximal element. We also find another condition under which the join of two subideals of a Lie algebra is a subideal (Theorem 1.5).

The second problem is to investigate the structure of Lie algebras with a certain chain condition on subideals using the notion of prime ideals and prime algebras (defined by analogy with associative rings). In particular we prove (Theorem 2.1) that when L is a Lie algebra over any field and $\mathfrak{X}$ is one of max-$\lhd^n$ $(n \geq 2)$, max-si, min-$\lhd^n$, min-si, $L \in \mathfrak{X}$ if and only if

(i) $\sigma(L)$ is a finite-dimensional soluble ideal of L.

(ii) $L/\sigma(L)$ is a subdirect sum of a finite number of prime algebras in $\mathfrak{X}$.

$\sigma(L)$ denotes a generalization of soluble radical.

Thirdly, we generalize the minimal condition on ideals, leading to a new class of quasi-Artinian algebras which possesses several of the main properties of min-$\lhd$. We prove (Theorems 3.2, 3.3) that the class of quasi-Artinian algebras is Q-closed and that a locally nilpotent quasi-Artinian Lie algebra is soluble.

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1. The join of subideals

It is well-known that the join of two subideals of a Lie algebra need not be a subideal (see [1, Lemma 2.1.11]). This raises the question of finding conditions under which the join is a subideal. The same question arises in group theory. Wielandt [9, Theorem 2.10.5] has shown that when $\{H_\lambda \mid \lambda \in \Lambda\}$ is a set of subnormal subgroups of a group G and J is their join, J is subnormal in G if and only if the set of subnormal subgroups of G lying in J contains a maximal member. We obtain a similar result for Lie algebras. In particular we prove an analogue