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Regular points for α-harmonic functions

Dedicated to Professor Makoto Ohtsuka on the occasion of his 60th birthday

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Let D be an open set in the n-dimensional Euclidean space R^n ($n \geqq 2$) and let $0<\alpha<2$. A boundary point $x_0 \in \partial D$ is said to be regular for D with respect to α-harmonic functions, or simply α-regular for D, if $\beta^{\alpha}_{CD}\varepsilon_{x_0}=\varepsilon_{x_0}$, where ε_{x_0} is the unit point measure at x_0 and β^{α}_{CD} denotes the balayage to the complement CD of D with respect to the α-potentials, i.e., the potentials of the kernel $|x|^{\alpha-n}$ (cf. [1; Chap. V]). Denote by D^{α}_{reg} the set of all α-regular points for D. In the problem section of [2], J. Veselý asks whether there exists a relatively compact open set D such that

(1) $$D^{\alpha}_{reg} \neq D^{\alpha'}_{reg} \quad \text{whenever} \quad \alpha \neq \alpha' \qquad (0 < \alpha, \alpha' < 2).$$

One of the purposes of this note is to answer this question, that is, to construct an open set D with property (1).

Through a communication with J. Veselý, the author learned that M. Kanda of Tsukuba University indicated him another solution to this problem which is more probabilistic.

Now, let us recall Wiener's criterion for α-regularity ([1; Theorem 5.2]):

Wiener's criterion: Let D be an open set and $E=CD$. Let $0<q<1$ and

$$E_k = E \cap \{x \in R^n;\ q^{k+1} \leqq |x-x_0| < q^k\}, \quad k = 1, 2, \ldots .$$

Then, $x_0 \in \partial D$ is α-regular ($0<\alpha<2$) if and only if

(2) $$\sum_{k=1}^{\infty} C_\alpha(E_k) q^{k(\alpha-n)} = \infty,$$

where C_α denotes the α-capacity (Riesz capacity of order α; cf. [1; Chap. II]). Now, we extend the definition of α-regularity for $0<\alpha<n$ by the above equality (2). In section 1, we shall construct an open set D for which (1) holds for $0<\alpha$, $\alpha'<n$.

By the definition of the α-capacity, we see easily that if $0<\alpha<\alpha'<n$, then

$$C_\alpha(F) \leqq C_{\alpha'}(F) d(F)^{\alpha'-\alpha}$$

for any bounded Borel set F, where $d(F)$ denotes the diameter of F. Therefore, in veiw of Wiener's criterion, $0<\alpha<\alpha'<n$ implies $D^{\alpha}_{reg} \subset D^{\alpha'}_{reg}$ for any open set D.