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On the q-dimension of a space of orderings and q-fans

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Let F be a formally real field, P a preordering and ρ a form over F. We shall say that a pair (ρ, P) is maximal if ρ is P-anisotropic and P is maximal among the preorderings over which ρ is anisotropic. For a given q-cone Q (cf. [7]) we shall define a preordering $P(Q)$ and show that, P being a preordering, (ρ, P) is a maximal pair for some form ρ if and only if P is of finite index and of the form $P=P(Q)$ for some q-cone Q; such a preordering will be called a q-fan in this paper.

The main purpose of this paper is to characterize a q-fan in terms of the q-dimension which is defined in §3, and give a reduction formula on q-dimensions (Theorem 3.6 and Theorem 3.9).

§1. Defintions and preliminaries

Throughout this paper, a field always means a formally real field. We denote by $\dot{F}$ the multiplicative group of F. For a multiplicative subgroup P of $\dot{F}$, P is said to be a preordering of F if P is additively closed and $\dot{F}^2 \subset P$. We denote by $X(F/P)$ the space of all orderings σ with $P(\sigma) \supset P$, where $P(\sigma)$ is the positive cone of σ. A valuation v of F is called a real valuation if its residue field is formally real. The objects: valuation ring, valuation ideal, group of units, residue field and group of values will be denoted by A, M, U, F_v and G respectively. A preordering P of F will be called compatible with a valuation v of F (or v is compatible with P) if $1+M \subset P$. If a preordering P of F is compatible with a valuation v, then $P \cap U$ is a union of cosets of M and $\bar{P}=\varphi(P \cap U)$ is a preordering of F_v, where φ is the canonical surjection: $A \to F_v$.

We shall say that two orderings $\sigma, \tau \in X(F/P)$ are connected in $X(F/P)$ if $\sigma=\tau$ or there exists a fan of index 8 which contains σ and τ, and we denote the relation by $\sigma \sim \tau$. It is known that the relation $\sim$ is an equivalence relation in $X(F/P)$ ([4], Theorem 4.7). Each equivalence class of this relation is called a connected component of $X(F/P)$. We say that a preordering P is connected if $X(F/P)$ is connected. We denote by $\mathrm{gr}(X(F/P))$ the translation group of $X(F/P)$ in the terminology of [4], namely $\mathrm{gr}(X(F/P))=\{\alpha \in \chi(F/P) \mid \alpha \cdot X(F/P)=X(F/P)\}$ where $\chi(F/P)=\mathrm{Hom}(\dot{F}/P, \{\pm 1\})$ is the character group of $\dot{F}/P$. For a preordering P of finite index, P is connected if and only if $\dim \dot{F}/P \geqq 3$ and $\dim \mathrm{gr}(X(F/P)) \geqq 1$.