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# Joins of weakly ascendant subalgebras of Lie algebras

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## Introduction

Amayo [2] proved that several classes of finitely generated Lie algebras are ascendantly coalescent, where a class $\mathfrak{X}$ of Lie algebras is ascendantly coalescent if in any Lie algebra the join of any pair of ascendant $\mathfrak{X}$-subalgebras is always an ascendant $\mathfrak{X}$-subalgebra. On the other hand, Tôgô [10] introduced the concept of weakly ascendant subalgebras of Lie algebras generalizing that of ascendant subalgebras. It might be hopeless to search classes $\mathfrak{X}$ such that in any Lie algebra the join of any pair of weakly ascendant $\mathfrak{X}$-subalgebras is always a weakly ascendant $\mathfrak{X}$-subalgebra, for there exists a Lie algebra in which the join of a certain pair of 1-dimensional weak subideals is not a weakly ascendant subalgebra and is non-abelian simple (cf. [4, Example 5.1]). However, in the recent papers [5] and [6] the author presented various classes of Lie algebras in which the join of any pair, or any family, of weak subideals (resp. subideals) is always a weak subideal (resp. a subideal). In this paper we shall investigate the class $\mathfrak{L}$(wasc) (resp. $\mathfrak{L}$(asc)) of Lie algebras in which the join of any pair of weakly ascendant subalgebras (resp. ascendant subalgebras) is always a weakly ascendant subalgebra (resp. an ascendant subalgebra), and the class $\mathfrak{L}^{\infty}$(wasc) (resp. $\mathfrak{L}^{\infty}$(asc)) of Lie algebras in which the join of any family of weakly ascendant subalgebras (resp. ascendant subalgebras) is always a weakly ascendant subalgebra (resp. an ascendant subalgebra).

Section 2 is devoted to investigating general properties of weakly ascendant subalgebras of Lie algebras. We shall show as generalizations of [2, Theorem 2.5] and [10, Theorem 4] that if $H$ wasc $L$ then $H/H_L \in \mathrm{L}\mathrm{R}\mathfrak{N} \cap \grave{\mathrm{E}}(\lhd)\mathfrak{A}$ (Theorem 2.2 (1)) and that if $H$ wasc $L$ and $H/H_{sL} \in \mathfrak{G}$ then $H \leq^{\omega} L$ (Theorem 2.2 (2)). Furthermore, we shall show that if $H \leq^{\rho} L$, $K \leq^{\sigma} L$ and $[H, K] \subseteq H$ then $H+K \leq^{\sigma\rho} L$ (Theorem 2.5).

In Section 3 we shall show that various classes are subclasses of $\mathfrak{L}(\Delta)$ or $\mathfrak{L}^{\infty}(\Delta)$, where $\Delta$ is any one of the relations wasc and asc. For example, the class $\mathfrak{D}$(wasc)$\mathfrak{A}$, which contains all hypercentral-by-abelian Lie algebras, is a subclass of $\mathfrak{L}$(wasc), and the classes $\mathfrak{D}$(wasc)$(\mathfrak{F} \cap \mathfrak{A}_1)$ and $\mathfrak{D}$(wasc)$\mathfrak{S}$(wasc) are subclasses of $\mathfrak{L}^{\infty}$(wasc) (Theorem 3.9). The class $\mathfrak{D}$(asc)$\mathfrak{A} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$, which contains all hypercentral-by-abelian Lie algebras, is a subclass of $\mathfrak{L}$(asc), and the classes $\mathfrak{D}$(asc)$(\mathfrak{F} \cap \mathfrak{A}_1) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\mathfrak{D}$(asc)$(\mathfrak{G} \cap \mathfrak{S})$, the latter of which contains all