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Picard principle for linear elliptic differential operators

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Take the punctured Euclidean unit n-ball Ω: $0<|x|<1$ $(x=(x_1,\cdots, x_n), n\geq 2)$. Throughout this paper we regard Ω as the subspace of the punctured Euclidean n-space M: $0<|x|<\infty$, so that the topological notions such as boundaries and closures etc. are considered relative to the "whole" space M. Hence $|x|=1$ is the boundary $\partial\Omega$ of Ω, $x=0$ is the ideal boundary of Ω, and the relative closure $\bar{\Omega}$ of Ω is $\Omega\cup\partial\Omega$. Consider an elliptic partial differential equation

$$Lu(x) \equiv \Delta u(x) + b(x)\cdot\nabla u(x) + c(x)u(x) = 0 \tag{1}$$

on Ω, where $\Delta=\sum_{i=1}^n \partial^2/\partial x_i^2$, $\nabla=(\partial/\partial x_1,\cdots, \partial/\partial_{x_n})$, $\cdot$ the inner product, and the vector field $b(x)=(b_1(x),\cdots, b_n(x))$ is of class C^2 on $\bar{\Omega}=\{0<|x|\leq 1\}$ and the function $c(x)$ of class C^1 on $\bar{\Omega}$ which may not be of constant sign. Thus the operator L is smooth on $\bar{\Omega}$ and especially on $\partial\Omega$: $|x|=1$, but may, and actually will, have singularities at $x=0$. We are interested in the class $\mathscr{P}$ of the non-negative solutions of (1) on Ω with vanishing boundary values on $\partial\Omega$. It is convenient to consider the normalized subclass $\mathscr{P}_1$ of $\mathscr{P}$ given by $\mathscr{P}_1=\{u\in\mathscr{P}$: $\int_{\partial\Omega}(\partial/\partial n_x)u(x)dS_x=1\}$, where $(\partial/\partial n_x)u(x)$ denotes the inner normal derivative of $u(x)$ at each point of $\partial\Omega$ whose existence is well known since $u(x)$ vanishes on $\partial\Omega$ (cf. e.g. Miranda [6]) and dS the surface element on $\partial\Omega$. Since $\mathscr{P}_1$ is convex, we can consider the set ex. $\mathscr{P}_1$ of extreme points of $\mathscr{P}_1$ and the cardinal number $\sharp(\text{ex.}\,\mathscr{P}_1)$ of ex. $\mathscr{P}_1$, which will be referred to as the *Picard dimension* of L at $x=0$, $\dim L$ in notation:

$$\dim L = \sharp(\text{ex.}\,\mathscr{P}_1). \tag{2}$$

We are particularly interested in the case $\dim L=1$. In this case we say, after Bouligand, that the *Picard principle* is valid for L at $x=0$. We will give a sufficient condition for its validity in terms of the orders of the growth of coefficients of L.

It can happen that $\dim L=0$. To prevent this trivial case we need to consider the existence of "Green's function" on Ω. For any point y fixed in Ω take a ball U: $|x-y|<a$ in Ω. If U is sufficiently small, then the Green's function (with respect to the Dirichlet problem) $g_U(x, y)$ on U for (1) with its pole y exists (cf. e.g. [6]). Consider a positive solution $u(x)$ of (1) on $\Omega-\{y\}$ satisfying the following two conditions: (i) $u(x)-g_U(x, y)$ is a solution of (1) on U; (ii) if $v(x)$