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Birational-integral extensions and differential modules

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Throughout this paper, a ring will mean a commutative Noetherian ring with identity.

Let R be a Noetherian domain and let $\bar{R}$ be the integral closure of R in its quotient field. An intermediate ring between R and $\bar{R}$ will be called a *birational-integral extension* of R. Let A be a birational-integral extension of R. We assume that A is a finite R-module. Let ${}^{+}_{A}R$ be the seminormalization of R in A. If ${}^{+}_{A}R=A$, then we say that the extension A/R is a *cuspidal type.*

In this paper, we shall prove that a cuspidal type extension is obtained by a finite chain of constant subrings of some derivations.

Let $C={}^{+}_{A}R$ and let I_C be the kernel of the canonical homomorphism

$$\Psi_C\colon C\otimes_R C \longrightarrow C.$$

Then I_C is generated by $\{\alpha\otimes 1-1\otimes\alpha/\alpha\in C\}$ and C/R is a cuspidal type extension. For any ring S, we put $S_{red}=S/nil(S)$ where $nil(S)$ denotes the nilradical of S. Let $\bar{\varphi}_A$ be a module-homomorphism of A to $(A\otimes_R A)_{red}$ over R defined by $\bar{\varphi}_A(\alpha)=\alpha\otimes 1-1\otimes\alpha \mod nil(A\otimes_R A)$. In [2], M. Manaresi proved that $\ker\bar{\varphi}_A={}^{w}_{A}R$ where ${}^{w}_{A}R$ is the weak normalization of R in A. In our situation, since $C={}^{+}_{C}R$, we have $C={}^{w}_{C}R$. By this result, each $\alpha\otimes 1-1\otimes\alpha$ $(\alpha\in C)$ is nilpotent and so I_C is nilpotent, say $I_C^{q+1}=(0)$ for some integer q. Then we see that the q-th order differential module $\Omega_R^q(C)=I_C/I_C^{q+1}$ of C over R is isomorphic to I_C and there exists the canonical q-th order derivation Δ_q of C over R to $\Omega_R^q(C)$ defined by $\Delta_q(\alpha)=\alpha\otimes 1-1\otimes\alpha$. We see that $\Delta_q^{-1}(0)$ is a subring of C containing R.

In the paper [1], J. Lipman introduced the following notion: For a ring S and a subring T of S, we say that

$${}^{*}_{S}T=\{\alpha\in S/\alpha\otimes 1=1\otimes\alpha \text{ in } S\otimes_T S\}$$

is the *strict closure* of T in S. If $T={}^{*}_{S}T$, then we say that T is *strictly closed* in S.

Using this notion, we have:

PROPOSITION 1. *Let R, C and Δ_q be as above, and let N be a $C\otimes_R C$-submodule of $\Omega_R^q(C)$ (for example, I_C^t, where t is an integer). Then $\Delta_q^{-1}(N)$ is strictly closed in C.*