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Zeta functions of Selberg's type associated with homogeneous vector bundles

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0. Introduction

Let G be a connected noncompact semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. Let $\tilde{M}$ be the symmetric space G/K. We endow $\tilde{M}$ with a G-invariant metric. We assume throughout this paper that $\operatorname{rank}(\tilde{M})=1$.

Let Γ be a discrete torsion-free subgroup of G such that the quotient $\Gamma\backslash G$ is compact. Γ acts on the symmetric space $\tilde{M}$ by left translations and the quotient space $\Gamma\backslash\tilde{M}$ is also compact. We give to the quotient manifold $\Gamma\backslash\tilde{M}$ which we will call $\bar{M}$, the push down Riemannian metric. Then $\bar{M}$ is the most general compact locally symmetric space of negative curvature. Also, the simply connected covering manifold of $\bar{M}$ is $\tilde{M}$, and we have $\pi_1(\bar{M})=\Gamma$.

Let T be a finite dimensional unitary representation of Γ on a vector space E_T with character χ_T. Since Γ is unimodular, there exists a G-invariant measure $d\dot{x}$ on the quotient space $\Gamma\backslash G$. We denote by $L^2(\Gamma\backslash G, T)$ the space of E_T valued measurable functions f on G such that (i) $f(\gamma x)=T(\gamma)f(x)$ for $\gamma\in\Gamma$, $x\in G$ and (ii) $\int_{\Gamma\backslash G}\|f(\dot{x})\|^2 d\dot{x}<\infty$. Since Γ is cocompact, the right regular representation $\pi_{\Gamma,T}$ of G on $L^2(\Gamma\backslash G, T)$ decomposes

$$\pi_{\Gamma,T} = \sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\pi$$

and $n_{\Gamma,T}(\pi)<\infty$ for any $\pi\in\hat{G}$. Here $\hat{G}$ stands for the set of all equivalence classes of irreducible unitary representations of G. Suppose that a function f is a C^∞ element of $L^2(\Gamma\backslash G, T)$ with compact support on G. Then the operator $\pi_{\Gamma,T}(f)=\int_G f(x)\pi_{\Gamma,T}(x)dx$ on $L^2(\Gamma\backslash G, T)$ is well defined and is of trace class. Therefore $\operatorname{tr}\pi_{\Gamma,T}(f)=\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi(f)$, where Θ_π denotes the character of the class π. On the other hand, we may compute a trace of $\pi_{\Gamma,T}(f)$ in a different manner by using the Selberg trace formula.

In this paper, applying a suitable function in $\mathscr{C}^1(G)$ to the trace formula, we will consider the generalization of the following results.

Let X be a compact Riemann surface of genus bigger than 2. Then $X=\Gamma\backslash H$ where $H=SL(2, \boldsymbol{R})/SO(2)$ is the upper half plane, and Γ is a discrete subgroup of