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Noetherian property of symbolic Rees algebras

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In the course of giving a counter-example to a problem of Zariski, D. Rees [6] proved the following theorem: Let $\mathfrak{p}$ be a prime ideal of a two-dimensional noetherian normal local domain with $ht(\mathfrak{p})=1$. If the graded ring $\oplus_{n\geqq 0}\mathfrak{p}^{(n)}$ is noetherian, then $\mathfrak{p}^{(d)}$ is a principal ideal for some $d\geqq 1$.

The aim of this note is to give a generalization of this theorem, which is stated as follows:

THEOREM. *Let $\mathfrak{p}$ be a prime ideal of a noetherian normal Nagata local domain R. Assume that $\dim(R/\mathfrak{p})=1$ and $R_{\mathfrak{p}}$ is regular. Then the graded ring $\oplus_{n\geqq 0}\mathfrak{p}^{(n)}$ is noetherian if and only if $\ell(\mathfrak{p}^{(d)})=\dim(R)-1$ for some $d\geqq 1$. Here we denote by $\ell(I)$ the analytic spread of an ideal I. (Concerning Nagata domains, see [3].)*

Throughout this paper, let R be a commutative ring and let I be an ideal of R. We denote by $S=R-Z_R(R/I)$ the set of R/I-regular elements of R, and for an R-module M, we put $M_I=M_S$. If R is a noetherian domain, then we have $R_I=\bigcap_{\mathfrak{p}\in\mathrm{Ass}_R(R/I)}R_{\mathfrak{p}}$. For an integer $n\geqq 0$, we define the *n-th symbolic power* $I^{(n)}$ of I by $I^{(n)}=I^nR_I\cap R=\{x\in R;\ tx\in R$ for some R/I-regular element $t\in R\}$.

PROPOSITION 1. (1) *$Z_R(R/I^{(n)})\subset Z_R(R/I)$ for all $n\geqq 1$.*

(2) *$I^{(1)}=I$, $\mathrm{rad}(I^{(n)})=\mathrm{rad}(I)$, $I^{(m)}I^{(n)}\subset I^{(m+n)}$ and $I^{(mn)}\subset I^{(m)(n)}$ for all $m, n\geqq 1$.*

(3) *Assume that R is noetherian and $\mathrm{Ass}_R(R/I)=\mathrm{Min}_R(R/I)$. Then $\mathrm{Ass}_R(R/I^{(n)})=\mathrm{Min}_R(R/I)$ for all $n\geqq 1$. In particular, $Z_R(R/I^{(n)})=Z_R(R/I)$ for all $n\geqq 1$. Also, we have $I^{(mn)}=I^{(m)(n)}$ for all $m, n\geqq 1$. Here we denote by $\mathrm{Min}_R(R/I)$ the set of minimal prime ideals of I.*

PROOF. (1) Assume that $t\in R$ is R/I-regular and $tx\in I^{(n)}$ for some $x\in R$. Then we have $s(tx)\in I^n$ for some R/I-regular element $s\in R$. Hence st is R/I-regular and $(st)x\in I^n$. This implies that $x\in I^{(n)}$.

(2) We prove the inclusion $I^{(mn)}\subset I^{(m)(n)}$. Take an element x of $I^{(mn)}$. Then for some R/I-regular element $t\in R$, we have $tx\in I^{mn}\subset I^{(m)n}$. Since t is $R/I^{(m)}$-regular by (1), we have $x\in I^{(m)(n)}$.

(3) If $\mathfrak{p}\in\mathrm{Ass}_R(R/I^{(n)})$, then $\mathfrak{p}\subset Z_R(R/I^{(n)})\subset Z_R(R/I)$. Hence $I\subset\mathfrak{p}\subset\mathfrak{q}$ for some $\mathfrak{q}\in\mathrm{Ass}_R(R/I)=\mathrm{Min}_R(R/I)$. Therefore we have $\mathfrak{p}=\mathfrak{q}\in\mathrm{Min}_R(R/I)$.