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Riesz's lemma and orthogonality in normed spaces

Dedicated to Professor Isao Miyadera on his 60th birthday

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Introduction

This paper is concerned with orthogonality in normed spaces and geometric structure of normed spaces as well as their dual spaces. The orthogonality problem is to discuss the existence and properties of elements that are orthogonal in an appropriate sense to a given closed subspace of a normed or Banach space, and problems of this kind are important in connection with the geometry of normed spaces. Our work is mainly devoted to two problems: The first aim is to seek natural notions of orhtogonality in general normed spaces; and the second purpose is to investigate various geometric properties of Banach spaces as well as those of incomplete normed spaces via the above notions of orthogonality.

Here we give a geometric interpretation of Riesz's Lemma in terms of duality theory of normed spaces and make an attempt to generalize the notion of Birkhoff orthogonality (see [2]) which is known as the most natural notion of orthogonality in general normed spaces. So-called Riesz's Lemma states that given a proper closed subspace M of a normed space X and a number $\varepsilon \in (0, 1)$ there is an element x_ε of X satisfying $\|x_\varepsilon\| = 1$ and $\mathrm{dist}(x_\varepsilon, M) \geqq 1-\varepsilon$. The standard use of Riesz's Lemma indicates that the Lemma is solely employed to find an element of norm 1 at a positive distance from a given proper closed subspace of a normed space, although the Lemma is directly related to the orthogonality problem in the following sense: If $\varepsilon = 0$ can be taken in the Lemma, then the associated unit vector x_0 turns out to be orthogonal to M in the sense of Birkhoff.

On the other hand, the James theorem and the Bishop-Phelps theorem, both of which are fundamental in Banach space theory, can be formulated as geometrical results concerning the orthogonality problem. First the former theorem asserts that a Banach space X is nonreflexive iff there is a hyperplane H in X such that none of the elements of X is orthogonal to H. This means that it is impossible to take $\varepsilon = 0$ in Riesz's Lemma if X is a nonreflexive Banach space. Now the latter theorem ensures a geometric property which compensates for this situation. Namely, the Bishop-Phelps theorem states that given a proper closed subspace M of a nonreflexive Banach space we can find a hyperplane H which is as close as we please to M and admits an element orthogonal to H. In other words, there is a sequence of unit vectors x_n, $n = 1, 2, 3, \ldots$, orthogonal respectively to hyperplanes