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When does LCM-stability ensure flatness at primes of depth one?

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Let R be a Noetherian integral domain and let M be an R-module. We say that M is LCM-stable over R if $(aR \cap bR)M = aM \cap bM$ for any elements $a, b \in R$ (cf. [1], [5]). F. Richman [4] proved that when A is an overring of R, that is, A is an intermediate ring between R and the field of quotients $K(R)$ of R, A is flat over R if and only if A is LCM-stable over R. The obstruction ideal $\mathscr{F}_R(A)$ (cf. [3]) has only depth one prime divisors. So if A is flat over R at primes of depth one, A is flat over R. Therefore the following question will arise:

When is the LCM-stable R-module M flat over R at each prime of depth one?

It is known that there is a module which is flat over a Noetherian normal domain R at each prime of depth one but is not LCM-stable over R. Our objective is to prove the following result which shows that the LCM-stable module over a Noetherian integral domain is not necessarily flat at primes of depth one:

Let R be a Noetherian integral domain and let M be a torsion-free, finite R-module. Assume that M is LCM-stable over R. Then M is reflexive if and only if $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for each $\mathfrak{p} \in Dp_1(R)$ $(:= \{\mathfrak{p} \in Spec\ R \mid \text{depth } R_{\mathfrak{p}} = 1\})$, i.e., M is flat over R at primes of depth one.

The following notation is fixed throughout this paper:

R denotes a (commutative) Noetherian integral doamin,
K the field of quotients of R,
$\bar{R}$ the integral closure of R in K and
M a non-zero torsion-free finite R-module.

We start with the following definition.

1. DEFINITION. Regard M as an R-submodule of $M_K := M \otimes_R K$. Define $\mathscr{R}(M)$ by

$$\mathscr{R}(M) := \{\alpha \in K \mid \alpha M \subseteq M\}.$$

2. ROPOSITION. *$\mathscr{R}(M)$ is an integral domain which contains R and is integral over R.*

PROOF. It is obvious that $\mathscr{R}(M)$ is an integral domain which contains R.